On Piterbarg's max-discretisation theorem for homogeneous Gaussian random fields*

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February 5, 2015

Abstract: Motivated by the papers of Piterbarg (2004) and Hüsler (2004), in this paper the asymptotic relation between the maximum of a continuous dependent homogeneous Gaussian random field and the maximum of this field sampled at discrete time points is studied. It is shown that, for the weakly dependent case, these two maxima are asymptotically independent, dependent and coincide when the grid of the discrete time points is a sparse grid, Pickands grid and dense grid, respectively, while for the strongly dependent case, these two maxima are asymptotically totally dependent if the grid of the discrete time points is sufficiently dense, and asymptotically dependent if the the grid points are sparse or Pickands grids.

Key Words: continuous time process, dependence, discrete time process, extreme values, homogeneous Gaussian random fields.

AMS Classification: Primary 60F05; secondary 60G15

1 Introduction

Let $\{X(t), t \geq 0\}$ be a stationary Gaussian process with mean 0, variance 1, correlation function $r(t)$ and continuous sample functions. The study on the limit distribution theory on the maximum of $\{X(t), t \geq 0\}$ up to time T: $M_T = \max\{X(t), 0 \leq t \leq T\}$ can be dated back to Pickands (1969). Assume that the correlation function $r(t)$ satisfies for some $\alpha \in (0, 2]$,

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0 \quad \text{and} \quad r(t) < 1 \quad \text{for} \quad t > 0 \tag{1}$$

and

$$r(t) \log t \to 0, \quad \text{as} \quad t \to \infty. \tag{2}$$

It is well known (see e.g. Pickands (1969), Leadbetter et al. 1983) that (1) and (2) imply the following classic limit relation

$$P\{a_T(M_T - b_T) \leq x\} \to \exp(-e^{-x}) \tag{3}$$

as $T \to \infty$, where

$$a_T = \sqrt{2 \log T}, \quad b_T = \sqrt{2 \log T} + \frac{\log[(2\pi)^{-1/2} \mathcal{H}_\alpha (2 \log T)^{-1/2+1/\alpha}]}{\sqrt{2 \log T}}.$$

*Research supported by National Science Foundation of China (No. 11326175), Natural Science Foundation of Zhejiang Province of China (No. LQ14A010012) and Research Start-up Foundation of Jiaxing University (No. 70512021).
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Here $\mathcal{H}_\alpha$ denotes Pickands constant, which is defined by $\mathcal{H}_\alpha = \lim_{\lambda\to\infty} \mathcal{H}_\alpha(\lambda)/\lambda$, with

$$\mathcal{H}_\alpha(\lambda) = \mathbb{E}\exp\left(\max_{t\in[0,\lambda]} \sqrt{2}B_{\alpha/2}(t) - t^\alpha\right)$$

and B_H is a fractional Brownian motion, that is a Gaussian zero mean process with stationary increments such that $\mathbb{E}B_H^2(t) = |t|^{2H}$. It is also well known that $0 < \mathcal{H}_\alpha < \infty$, see e.g. Pickands (1969), Leadbetter et al. (1983), Piterbarg (1996).

The extensions of the classic result (3) to more general cases, such as for non-stationary case, strongly dependent case, can be found in Mittal and Ylvisaker (1975), McCormick (1980), McCormick and Qi (2000), Hülser (1990), Konstant and Piterbarg (1993), Seleznjev (1991, 1996), Hülser (1999), Hülser et al. (2003), Tan et al. (2012) and among others.

In applied fields, however, the classic result (3) can not be used directly, since the available samples are discrete. Usually, simulation techniques are applied to derive results for continuous process when they can not be derived with mathematical analytic tools. Simulations of such processes are performed for discrete time-grids, while the results should be interpreted in the context of continuous time. Therefore, it is crucial to investigate the relation between the extremes of the continuous process and the extremes of the discrete process.

Piterbarg (2004) first studied the asymptotic relation between M_T and the maximum of the discrete version $M_T^\delta = \max\{X(k\delta), 0 \le k\delta \le T\}$ for some $\delta = \delta(T) > 0, k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Following Piterbarg (2004), we consider uniform grids $\mathfrak{R} = \mathfrak{R}(\delta) = \{k\delta : k \in \mathbb{N}\}$, $\delta > 0$. A grid is called sparse if δ is such that

$$\delta(2\log T)^{1/\alpha} \to D$$

with $D = \infty$. If $D \in (0, \infty)$, the grid is a Pickands grid, and if $D = 0$, the grid is dense.

For the stationary Gaussian processes, Piterbarg (2004) first showed that the maximum M_T^δ of discrete time points and the maximum M_T of the continuous time points can be asymptotically independent, dependent or totally dependent if the grid is a sparse, a Pickands or a dense grid, respectively. This type of results are called Piterbarg's max-discretisation theorems in the literature, see eg. Tan and Hashorva (2014a).

Based on the results of Hüsler (1990), Piterbarg's max-discretisation theorems were extended by Hüsler (2004) to a class of locally stationary Gaussian processes which was introduced by Berman (1974). Other related results such as for the storage process with fractional Brownian motion as input and stationary non-Gaussian case can be found in Hüsler and Piterbarg (2004) and Turkman (2012), respectively. The recent contributions Tan and Wang (2013) and Tan and Tang (2014) present Piterbarg's max-discretisation theorem for strongly dependent stationary Gaussian processes. The Piterbarg's max-discretisation theorems for multivariate Gaussian processes can be found in Tan and Hashorva (2014b) and their improvement to different grids can be found in Tan and Hashorva (2015).

The Piterbarg's max-discretisation theorems for Gaussian processes have been studied extensively under different conditions in the past, but it is far from complete. In this paper, we are interested in the similar problems for the Gaussian random fields. It is well known that Gaussian random fields play a very important role in many applied sciences, such as in image analysis, atmospheric sciences, geostatistics, neuroimaging, astrophysics, oceanography, hydrology and agriculture, among others, see eg. Adler and Taylor (2007) for details. Extremes and their limit properties are particularly important in these applications, see, for instance, Azaïs and Wschebor (2009).

The paper is organized as follows: In Section 2, we present the main results for weakly and strongly dependent Gaussian fields. Section 3 gives the proofs. Some technical auxiliary results are presented in Sections 4 and 5. Let ϕ and Ψ denote the density function and tail distribution function of a standard normal variable.

2 Main results

Denote the set of all real numbers by $\mathbb{R}$ and let $\mathbb{R}^d$ be d-dimensions product space of $\mathbb{R}$, where $d \ge 2$. In this paper, we only consider the case of $d = 2$ since it is notationally simplest and the results for higher dimensions follow analogouss arguments. Here the operations with the vectors are meant component-wise. For instance for two

vectors $\mathbf{t} = (t_1, t_2)$ and $\mathbf{s} = (s_1, s_2)$, $\mathbf{s} \leq \mathbf{t}$, $\mathbf{t} - \mathbf{s}$ and $\mathbf{st}$ mean $s_i \leq t_i$, $i = 1, 2$, $(t_1 - s_1, t_2 - s_2)$ and $(s_1 t_1, s_2 t_2)$, respectively. $\mathbf{T} \to \infty$ means $T_i \to \infty$, $i = 1, 2$. Let $\mathbf{I_T} = \{\mathbf{t} \in \mathbb{R}^2 : 0 \leq t_i \leq T_i, i = 1, 2\}$. Let $\{X(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\}$ denote a homogeneous Gaussian field with covariance function

$$r(\mathbf{t}) = \mathbb{C}ov(X(\mathbf{t}), X(\mathbf{0})).$$

In this paper we assume that the covariance function satisfies the following conditions:

A1: $r(\mathbf{t}) = 1 - |t_1|^{\alpha_1} - |t_2|^{\alpha_2} + o(|t_1|^{\alpha_1} + |t_2|^{\alpha_2})$ as $\mathbf{t} \to 0$ with $\alpha_i \in (0, 2]$;

A2: $r(\mathbf{t}) < 1$ for $\mathbf{t} \neq \mathbf{0}$;

A3: $\lim_{\mathbf{t} \to \infty} r(\mathbf{t}) \log(t_1 t_2) = r \in [0, \infty)$ and both $r(0, t_2) \log t_2$ and $r(t_1, 0) \log t_1$ are bounded.

Throughout the paper, for any set $\mathbf{E} \subset \mathbb{R}^2$ and $\mathbf{k} \in \mathbb{N}^2$, define

$$M_{\mathbf{E}} = \max_{\mathbf{t} \in \mathbf{E}} X(\mathbf{t}) = \max\{X(\mathbf{t}), \mathbf{t} \in \mathbf{E}\}, \quad M_{\mathbf{E}}^{\mathbf{p}} = \max_{\mathbf{t} \in \mathbf{E} \cap \mathfrak{R}(p_1) \times \mathfrak{R}(p_2)} X(\mathbf{t}) = \max\{X(\mathbf{kp}), \mathbf{kp} \in \mathbf{E}\},$$

where $\mathfrak{R}(p_i) = \{kp_i, k \in \mathbb{N}\}$, $i = 1, 2$, are uniform grids. If $\mathbf{E} = \mathbf{I_T}$, we write the above two maxima for simplicity as $M_{\mathbf{T}}$ and $M_{\mathbf{T}}^{\mathbf{p}}$, respectively. For dealing with the multivariate case, we redefine the uniform grids $\mathfrak{R}(p_i) = \{kp_i, k \in \mathbb{N}\}$, $i = 1, 2$ as following. The grid $\mathfrak{R}(p_i)$ is called sparse if $p_i = p_i(\mathbf{T})$ is such that

$$p_i (2 \log T_1 T_2)^{1/\alpha_i} \to D_i, \quad i = 1, 2$$

with $D_i = \infty$. If $D_i \in (0, \infty)$, the grid is a Pickands grid, and if $D_i = 0$, the grid is dense.

Under conditions **A1** and **A2**, Theorem 7.1 of Piterbarg (1996) showed that for any fixed $\mathbf{h} > \mathbf{0}$

$$P\left(\max_{\mathbf{t} \in \mathbf{I_h}} X(\mathbf{t}) > u\right) = \mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2} h_1 h_2 u^{2/\alpha_1 + 2/\alpha_2} \Psi(u)(1 + o(1)), \tag{4}$$

as $u \to \infty$, where $\mathcal{H}_{\alpha_i}$, $i = 1, 2$ are the Pickands constants. This exact asymptotic plays crucial role in deriving the Gumbel law and also will be used in the proofs of our main results.

Now, we state our main results which extend the existing results (including Piterbarg (2004) and Tan and Wang (2013)) to Gaussian random fields. The extensions are, however, nontrivial in that asymptotic relation between two Gaussian fields is more complicated than that of Gaussian processes. This can be seen from the proof that follows.

Theorem 2.1. *Let $\{X(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\}$ be a homogeneous Gaussian field with covariance function $r(\mathbf{t})$ satisfying **A1**, **A2** and **A3**. Then for any sparse grids $\mathfrak{R}(p_i)$, $i = 1, 2$,*

$$\begin{aligned} &P\left\{a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \leq x, a_{\mathbf{T}}(M_{\mathbf{T}}^{\mathbf{p}} - b_{\mathbf{T}}^{\mathbf{p}}) \leq y\right\} \\ &\quad \longrightarrow \int_{-\infty}^{+\infty} \exp\left(-\left(e^{-x-r+\sqrt{2r}z} + e^{-y-r+\sqrt{2r}z}\right)\right) \phi(z) dz \end{aligned} \tag{5}$$

as $\mathbf{T} \to \infty$, where

$$a_{\mathbf{T}} = \sqrt{2 \log T_1 T_2}, \quad b_{\mathbf{T}} = a_{\mathbf{T}} + a_{\mathbf{T}}^{-1} \log\left((2\pi)^{-1/2} \mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2} (a_{\mathbf{T}})^{1/\alpha_1 + 1/\alpha_2 - 1/2}\right)$$

and

$$b_{\mathbf{T}}^{\mathbf{p}} = a_{\mathbf{T}} + a_{\mathbf{T}}^{-1} \log\left((2\pi)^{-1/2} p_1^{-1} p_2^{-1} (a_{\mathbf{T}})^{-1/2}\right).$$

As the special case, we can obtain the limit distribution of the maximum for a homogeneous Gaussian random field, which will be used in the proof of Theorem 2.3.

Corollary 2.1. *Let $\{X(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\}$ be a homogeneous Gaussian field with covariance function $r(\mathbf{t})$ satisfying **A1**, **A2** and **A3**. Then for any $x \in \mathbb{R}$,*

$$P\left\{a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \leq x\right\} \longrightarrow \int_{-\infty}^{+\infty} \exp\left(-e^{-x-r+\sqrt{2r}z}\right) \phi(z) dz \tag{6}$$

as $\mathbf{T} \to \infty$.

Remark 2.2. A similar result as Corollary 2.1 can also be derived from Theorem 15.2 in Chapter 4 of Piterbarg (1996), where the author dealt with the limit properties of uncrossing point processes under some slight different conditions.

Before presenting the result for Pickands grids, we introduce the following Pickands type constants. For $a > 0$, define,

$$\mathcal{H}_{a,\alpha}(\lambda) = \mathbb{E}\exp\left(\max_{ka\in[0,\lambda]} \sqrt{2}B_{\alpha/2}(ka) - (ka)^{\alpha}\right),$$

we have (see Leadbetter et al. 1983),

$$\mathcal{H}_{a,\alpha} = \lim_{\lambda\to\infty} \frac{\mathcal{H}_{a,\alpha}(\lambda)}{\lambda} \in (0,\infty).$$

For any $\mathbf{d} > \mathbf{0}$, define

$$\mathcal{H}^{x,y}_{\mathbf{d},\alpha_1,\alpha_2}(\lambda_1,\lambda_2) = \int_{-\infty}^{+\infty} e^{s} P\Bigg(\max_{(t_1,t_2)\in[0,\lambda_1]\times[0,\lambda_2]} \sqrt{2}\chi(t_1,t_2) > s+x,$$
$$\max_{(k_1d_1,k_2d_2)\in[0,\lambda_1]\times[0,\lambda_2]} \sqrt{2}\chi(k_1d_1,k_2d_2) > s+y\Bigg)\, ds,$$

where

$$\chi(t_1,t_2) = B^{(1)}_{\alpha_1/2}(t_1) + B^{(2)}_{\alpha_2/2}(t_2) - |t_1|^{\alpha_1} - |t_2|^{\alpha_2}$$

and $B^{(1)}_{\alpha_1/2}(\cdot)$, $B^{(2)}_{\alpha_2/2}(\cdot)$ are two independent fractional Brownian motions.

Theorem 2.2. *Let $\{X(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\}$ be a homogeneous Gaussian field with covariance function $r(\mathbf{t})$ satisfying* ***A1****,* ***A2*** *and* ***A3****. Then for any Pickands grids $\mathfrak{R}(p_i) = \mathfrak{R}(a_i(2\log T_1T_2)^{-1/\alpha_i})$ with $a_i > 0$, $i = 1,2$, the following limit exists,*

$$\mathcal{H}^{x,y}_{\mathbf{a},\alpha_1,\alpha_2} := \lim_{\substack{\lambda_1\to\infty\\ \lambda_2\to\infty}} \mathcal{H}^{x,y}_{\mathbf{a},\alpha_1,\alpha_2}(\lambda_1,\lambda_2)/\lambda_1\lambda_2 \in (0,\infty)$$

and

$$P\left\{a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \leq x, a_{\mathbf{T}}(M^{\mathbf{p}}_{\mathbf{T}} - b_{\mathbf{a},\mathbf{T}}) \leq y\right\}$$
$$\longrightarrow \int_{-\infty}^{+\infty} \exp\left(-\left(e^{-x-r+\sqrt{2r}z} + e^{-y-r+\sqrt{2r}z} - \mathcal{H}^{\log(\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2})+x,\log(\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2})+y}_{\mathbf{a},\alpha_1,\alpha_2} e^{-r+\sqrt{2r}z}\right)\right)\phi(z)dz \quad (7)$$

as $\mathbf{T} \to \infty$, *where*

$$b_{\mathbf{a},\mathbf{T}} = a_{\mathbf{T}} + a^{-1}_{\mathbf{T}} \log\left((2\pi)^{-1/2}\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2}(a_{\mathbf{T}})^{1/\alpha_1+1/\alpha_2-1/2}\right).$$

Theorem 2.3. *Let $\{X(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\}$ be a homogeneous Gaussian field with covariance function $r(\mathbf{t})$ satisfying* ***A1****,* ***A2*** *and* ***A3****. Then for any dense grids $\mathfrak{R}(p_i)$, $i = 1,2$,*

$$P\left\{a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \leq x, a_{\mathbf{T}}(M^{\mathbf{p}}_{\mathbf{T}} - b_{\mathbf{T}}) \leq y\right\} \longrightarrow \int_{-\infty}^{+\infty} \exp\left(-e^{-\min(x,y)-r+\sqrt{2rz}}\right)\phi(z)dz \quad (8)$$

as $\mathbf{T} \to \infty$.

Remark 2.2. i). In the literature, the Gaussian field $X(\mathbf{t})$ with correlation satisfying $\lim_{\mathbf{t}\to\infty} r(\mathbf{t})\log(t_1t_2) = r \in [0,\infty)$ is called weakly and strongly dependent for $r = 0$ and $r > 0$, respectively, see eg., Mittal and Ylvisaker (1975). Theorems 2.1-2.3 show that for the weakly dependent case the two maxima are asymptotically independent, dependent and coincide when the grid of the discrete time points is a sparse grid, Pickands grid and dense grid, respectively. For the strongly dependent case, the asymptotic independence between the two maxima does not hold anymore because of the strong dependence of $X(\mathbf{t})$. However, in this case these two maxima are asymptotically totally dependent if the grid of the discrete time points is sufficiently dense, and asymptotically dependent if the the grid points are sparse or Pickands grids.

ii). If $T_1 = O(T_2)$, as $\mathbf{T} \to \infty$, then the condition that $r(0,t_2)\log t_2$ and $r(t_1,0)\log t_1$ are bounded in Assumption

A3 can be omitted. Noting that Assumption **A3** is only used in the proof of Lemma 3.3, it is easy to check this point from the bounds of $S_{\mathbf{T},22}$ and $M_{\mathbf{T},22}$ in the proofs of Lemma B1 and B3, respectively. However, if $T_1 = o(T_2)$, as $\mathbf{T} \to \infty$, then Assumption **A3** can be weakened as: $\lim_{\mathbf{t}\to\infty} r(\mathbf{t}) \log(t_1 t_2) = r \in [0, \infty)$ and $r(0, t_2) \log t_2$ is bounded. A similar statement holds also for the case $T_2 = o(T_1)$.

3 Proofs

First, define $\rho(\mathbf{T}) = r/\log(T_1 T_2)$ and let $a > b$ be constants which will be determined in the proof of Lemma 3.3. Following Piterbarg (2004), divide $[0, T_i]$ into intervals with length T_i^a alternating with shorter intervals with length T_i^b, $i = 1, 2$. Note that the numbers of the long intervals is at most $n_i = \lfloor T_i/(T_i^a + T_i^b) \rfloor$, where $\lfloor x \rfloor$ denote the integral parts of x. Let $\mathbf{O_i} = [(i_1-1)(T_1^a+T_1^b), (i_1-1)(T_1^a+T_1^b)+T_1^a] \times [(i_2-1)(T_2^a+T_2^b), (i_2-1)(T_2^a+T_2^b)+T_2^a]$, $\mathbf{i} = \mathbf{1}, \cdots, \mathbf{n}$ and $\mathbf{O} = \cup_{\mathbf{i}} \mathbf{O_i}$. We will show blow that the remaining area $\mathbf{I_T} \backslash \mathbf{O}$ plays no role in our consideration. Let $\{X_{\mathbf{i}}(\mathbf{t}), \mathbf{t} \ge \mathbf{0}\}$, $\mathbf{i} \ge \mathbf{1}$ be independent copies of $\{X(\mathbf{t}), \mathbf{t} \ge \mathbf{0}\}$ and $\{\eta(\mathbf{t}), \mathbf{t} \ge \mathbf{0}\}$ be such that $\eta(\mathbf{t}) = X_{\mathbf{i}}(\mathbf{t})$ for $\mathbf{t} \in \mathbf{E_i} := [(i_1-1)(T_1^a+T_1^b), i_1(T_1^a+T_1^b)) \times [(i_2-1)(T_2^a+T_2^b), i_2(T_2^a+T_2^b))$, $\mathbf{i} = \mathbf{1}, \cdots, \mathbf{n}$. Define

$$\xi_{\mathbf{T}}(\mathbf{t}) = \left(1 - \rho(\mathbf{T})\right)^{1/2} \eta(\mathbf{t}) + \rho^{1/2}(\mathbf{T}) U, \quad \mathbf{t} \in \mathbf{I_T},$$

where U is a standard normal variable independent of $\{\eta(\mathbf{t}), \mathbf{t} \ge \mathbf{0}\}$. Denote by $\varrho(\mathbf{s}, \mathbf{t})$ the covariance function of $\{\xi_{\mathbf{T}}(\mathbf{t}), \mathbf{t} \in \mathbf{I_T}\}$. It is easy to check that

$$\varrho(\mathbf{s}, \mathbf{t}) = \begin{cases} r(\mathbf{t}, \mathbf{s}) + (1 - r(\mathbf{t}, \mathbf{s}))\rho(\mathbf{T}), & \mathbf{s} \in \mathbf{E_i}, \mathbf{t} \in \mathbf{E_j}, \mathbf{i} = \mathbf{j}; \\ \rho(\mathbf{T}), & \mathbf{s} \in \mathbf{E_i}, \mathbf{t} \in \mathbf{E_j}, \mathbf{i} \neq \mathbf{j}. \end{cases}$$

The proofs of our main results rely on the following Lemmas. In the sequel, C shall denote positive constant whose values may vary from place to place.

Lemma 3.1. *Suppose that the grids $\mathfrak{R}(p_i)$, $i = 1, 2$ are sparse grids or Pickands grids. For any $B > 0$, we have for all $x, y \in [-B, B]$,*

$$\Big| P\left\{a_{\mathbf{T}}\left(M_{\mathbf{T}} - b_{\mathbf{T}}\right) \le x, a_{\mathbf{T}}\left(M_{\mathbf{T}}^{\mathbf{p}} - b_{\mathbf{T}}'\right) \le y\right\} - P\left\{a_{\mathbf{T}}\left(\max_{\mathbf{t}\in\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}\right) \le x, a_{\mathbf{T}}\left(\max_{\mathbf{t}\in\mathfrak{R}(\delta_1)\times\mathfrak{R}(\delta_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}'\right) \le y\right\} \Big| \to 0$$

as $\mathbf{T} \to \infty$, where $b_{\mathbf{T}}' = b_{\mathbf{T}}^{\mathbf{p}}$ for sparse grids and $b_{\mathbf{T}}' = b_{\mathbf{a},\mathbf{T}}$ for Pickands grids.

Proof: The proof is similar to that of Lemma 6 of Piterbarg (2004). Clearly, we have

$$\begin{aligned} &\Big| P\left\{a_{\mathbf{T}}\left(M_{\mathbf{T}} - b_{\mathbf{T}}\right) \le x, a_{\mathbf{T}}\left(M_{\mathbf{T}}^{\mathbf{p}} - b_{\mathbf{T}}'\right) \le y\right\} \\ &\quad - P\left\{a_{\mathbf{T}}\left(\max_{\mathbf{t}\in\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}\right) \le x, a_{\mathbf{T}}\left(\max_{\mathbf{t}\in\mathfrak{R}(\delta_1)\times\mathfrak{R}(\delta_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}'\right) \le y\right\} \Big| \\ &\quad \le P\left\{\max_{\mathbf{t}\in\mathbf{I_T}\backslash\mathbf{O}} X(\mathbf{t}) > b_{\mathbf{T}} + x/a_{\mathbf{T}}\right\} + P\left\{\max_{\mathbf{t}\in\mathfrak{R}(\delta_1)\times\mathfrak{R}(\delta_2)\cap\mathbf{I_T}\backslash\mathbf{O}} X(\mathbf{t}) > b_{\mathbf{T}}' + y/a_{\mathbf{T}}\right\} \end{aligned} \tag{9}$$

By Theorem 7.2 of Piterbarg (1996) (denote by $mes(\cdot)$ the Lebesgue measure)

$$\begin{aligned} P\left\{\max_{\mathbf{t}\in\mathbf{I_T}\backslash\mathbf{O}} X(\mathbf{t}) > b_{\mathbf{T}} + x/a_{\mathbf{T}}\right\} &= O(1) mes(\mathbf{I_T}\backslash\mathbf{O})(b_{\mathbf{T}} + x/a_{\mathbf{T}})^{2/\alpha_1 + 2/\alpha_2} \Psi(b_{\mathbf{T}} + x/a_{\mathbf{T}}) \\ &= O(1) \frac{mes(\mathbf{I_T}\backslash\mathbf{O})}{T_1 T_2} \\ &\le O(1) \frac{n_1 T_1^b (T_2^a + T_2^b) + n_2 T_2^b (T_1^a + T_1^b)}{T_1 T_2} \to 0 \end{aligned}$$

as $\mathbf{T} \to \infty$, by the choice of $a_{\mathbf{T}}$ and $b_{\mathbf{T}}$. In light of (16) and (22) in the Appendix for a sparse grid and Pickands grid, respectively, we can get the same estimation for the second probability in the right-hand side of (9), hence the proof is complete. $\square$

For the proofs we need also the following auxiliary grids $\mathfrak{R}(q_i)$ with $q_i = \gamma_i(2\log T_1T_2)^{-1/\alpha_i}$ and $\gamma_i > 0$, $i = 1, 2$.

Lemma 3.2. *Suppose that the grids $\mathfrak{R}(p_i)$, $i = 1, 2$ are sparse grids or Pickands grids. For any $B > 0$, we have for all $x, y \in [-B, B]$*

$$\left| P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right.$$
$$\left. - P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(q_1)\times\mathfrak{R}(q_2)\cap\mathbf{O}} X(\mathbf{t}) - b_T\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right| \to 0$$

as $\mathbf{T} \to \infty$ and $\gamma_i \downarrow 0$, where $b_{\mathbf{T}}' = b_{\mathbf{T}}^{\mathbf{p}}$ for sparse grids and $b_{\mathbf{T}}' = b_{\mathbf{a},\mathbf{T}}$ for Pickands grids.

Proof: It follows from Lemma A4. □

The following lemma plays a crucial role in the proofs of Theorems 2.1 and 2.2.

Lemma 3.3. *Suppose that the grids $\mathfrak{R}(p_i)$, $i = 1, 2$ are sparse grids or Pickands grids. For any $B > 0$ we have for all $x, y \in [-B, B]$,*

$$\left| P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(q_1)\times\mathfrak{R}(q_2)\cap\mathbf{O}} X(\mathbf{t}) - b_T\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right.$$
$$\left. - P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(q_1)\times\mathfrak{R}(q_2)\cap\mathbf{O}} \xi_{\mathbf{T}}(\mathbf{t}) - b_{\mathbf{T}}\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} \xi_{\mathbf{T}}(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right| \to 0$$

as $\mathbf{T} \to \infty$, where $b_{\mathbf{T}}' = b_{\mathbf{T}}^{\mathbf{p}}$ for sparse grids and $b_{\mathbf{T}}' = b_{\mathbf{a},\mathbf{T}}$ for Pickands grids.

Proof: For the sake of simplicity, let $u_{\mathbf{T}} = b_{\mathbf{T}} + x/a_{\mathbf{T}}$, $u_{\mathbf{T}}' = b_{\mathbf{T}}' + y/a_{\mathbf{T}}$. Using the Normal Comparison Lemma (see eg. Leadbetter et al. (1983) and Piterbarg (1996)), we have

$$\left| P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(q_1)\times\mathfrak{R}(q_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} X(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right.$$
$$\left. - P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(q_1)\times\mathfrak{R}(q_2)\cap\mathbf{O}} \xi_{\mathbf{T}}(\mathbf{t}) - b_{\mathbf{T}}\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} \xi_{\mathbf{T}}(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right|$$
$$\le \sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j}\\ \mathbf{kq}\ne\mathbf{lq},1\le\mathbf{i},\mathbf{j}\le\mathbf{n}}} |r(\mathbf{kq},\mathbf{lq}) - \varrho(\mathbf{kq},\mathbf{lq})| \int_0^1 \frac{1}{\sqrt{1 - r^{(h)}(\mathbf{kq},\mathbf{lq})}} \exp\left(-\frac{u_{\mathbf{T}}^2}{1 + r^{(h)}(\mathbf{kq},\mathbf{lq})}\right) dh$$
$$+ \sum_{\substack{\mathbf{kp}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j}\\ \mathbf{kp}\ne\mathbf{lp},1\le\mathbf{i},\mathbf{j}\le\mathbf{n}}} |r(\mathbf{kp},\mathbf{lp}) - \varrho(\mathbf{kp},\mathbf{lp})| \int_0^1 \frac{1}{\sqrt{1 - r^{(h)}(\mathbf{kp},\mathbf{lp})}} \exp\left(-\frac{u_{\mathbf{T}}'^2}{1 + r^{(h)}(\mathbf{kp},\mathbf{lp})}\right) dh$$
$$+ \sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j}\\ \mathbf{kq}\ne\mathbf{lp},1\le\mathbf{i},\mathbf{j}\le\mathbf{n}}} |r(\mathbf{kq},\mathbf{lp}) - \varrho(\mathbf{kq},\mathbf{lp})| \int_0^1 \frac{1}{\sqrt{1 - r^{(h)}(\mathbf{kq},\mathbf{lp})}} \exp\left(-\frac{u_{\mathbf{T}}'^2 + u_{\mathbf{T}}^2}{2(1 + r^{(h)}(\mathbf{kq},\mathbf{lp}))}\right) dh,$$

where $r^{(h)}(\mathbf{kq},\mathbf{lq}) = hr(\mathbf{kq},\mathbf{lq}) + (1-h)\varrho(\mathbf{kq},\mathbf{lq})$. Now, the lemma follows from Lemmas B1-B3 in the Appendix B. □

Lemma 3.4. *Suppose that the grids $\mathfrak{R}(p_i)$, $i = 1, 2$ are sparse grids or Pickands grids. For any $B > 0$ we have for all $x, y \in [-B, B]$ and the grids $\mathfrak{R}(q_i)$ with $q_i = \gamma_i(2\log T_1T_2)^{-1/\alpha_i}$ and $\gamma_i > 0$, $i = 1, 2$*

$$\left| P\left\{ a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(q_1)\times\mathfrak{R}(q_2)\cap\mathbf{O}} \xi_{\mathbf{T}}(\mathbf{t}) - b_{\mathbf{T}}\big) \le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O}} \xi_{\mathbf{T}}(\mathbf{t}) - b_{\mathbf{T}}'\big) \le y \right\} \right.$$
$$\left. - \int_{-\infty}^{+\infty} \prod_{i_1=1}^{n_1} \prod_{i_2=1}^{n_2} P\left\{ \max_{\mathbf{t}\in\mathbf{O_i}} \eta(\mathbf{t}) \le u_{\mathbf{T}}^*, \max_{\mathbf{t}\in\mathbf{O_i}} \eta(\mathbf{t}) \le u_{\mathbf{T}}^{*'} \right\} \phi(z) dz \right| \to 0,$$

as $\gamma_i \downarrow 0$, where

$$u_{\mathbf{T}}^* := \frac{b_{\mathbf{T}} + x/a_{\mathbf{T}} - \rho^{1/2}(\mathbf{T})z}{(1 - \rho(\mathbf{T}))^{1/2}} = \frac{x + r - \sqrt{2r}z}{a_{\mathbf{T}}} + b_{\mathbf{T}} + o(a_{\mathbf{T}}^{-1}), \tag{10}$$

and

$$u_{\mathbf{T}}^{*'} := \frac{b_{\mathbf{T}}' + y/a_{\mathbf{T}} - \rho^{1/2}(\mathbf{T})z}{(1 - \rho(\mathbf{T}))^{1/2}} = \frac{y + r - \sqrt{2r}z}{a_{\mathbf{T}}} + b_{\mathbf{T}}' + o(a_{\mathbf{T}}^{-1}) \tag{11}$$

with $b_{\mathbf{T}}^{'}=b_{\mathbf{T}}^{\mathbf{p}}$ for sparse grids and $b_{\mathbf{T}}^{'}=b_{\mathbf{a},\mathbf{T}}$ for Pickands grids.

Proof: First, by the definition of $\{\xi_{\mathbf{T}}(\mathbf{t}),\mathbf{0}\le\mathbf{t}\le\mathbf{T}\}$, we have

$$\begin{aligned}
&P\left\{a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\Re(q_1)\times\Re(q_2)\cap\mathbf{O}}\xi_{\mathbf{T}}(\mathbf{t})-b_{\mathbf{T}}\big)\le x, a_{\mathbf{T}}\big(\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O}}\xi_{\mathbf{T}}(\mathbf{t})-b_{\mathbf{T}}^{'}\big)\le y\right\}\\
&=\int_{-\infty}^{+\infty}P\left\{\max_{\mathbf{t}\in\Re(q_1)\times\Re(q_2)\cap\mathbf{O}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\phi(z)dz\\
&=\int_{-\infty}^{+\infty}\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}P\left\{\max_{\mathbf{t}\in\Re(q_1)\times\Re(q_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{t\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\phi(z)dz. \qquad (12)
\end{aligned}$$

As for the discrete case, see page 137 on Leadbetter et al. (1983), a direct calculation leads to

$$u_{\mathbf{T}}^{*} = \frac{x+r-\sqrt{2r}z}{a_{\mathbf{T}}}+b_{\mathbf{T}}+o(a_{\mathbf{T}}^{-1})$$

and

$$u_{\mathbf{T}}^{*'} = \frac{y+r-\sqrt{2r}z}{a_{\mathbf{T}}}+b_{\mathbf{T}}^{'}+o(a_{\mathbf{T}}^{-1}).$$

Next, by Lemma A4 and the dominated convergence theorem, we have

$$\begin{aligned}
&\Bigg|\int_{-\infty}^{+\infty}\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}P\left\{\max_{\mathbf{t}\in\Re(q_1)\times\Re(q_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\phi(z)dz\\
&\quad-\int_{-\infty}^{+\infty}\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}P\left\{\max_{\mathbf{t}\in\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\phi(z)dz\Bigg|\to 0 \qquad (13)
\end{aligned}$$

as $\gamma_i\downarrow 0$. Lemma 3.4 now follows by combining (12) with (13). □

Proof of Theorem 2.1. From Lemmas 3.1-3.4, we known that in order to prove Theorem 2.1, it suffice to show that

$$\begin{aligned}
&\Bigg|\int_{-\infty}^{+\infty}\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}P\left\{\max_{\mathbf{t}\in\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\phi(z)dz\\
&\quad-\int_{-\infty}^{+\infty}\exp\left(-\big(e^{-x-r+\sqrt{2r}z}+e^{-y-r+\sqrt{2r}z}\big)\right)\phi(z)dz\Bigg|\to 0
\end{aligned}$$

as $\mathbf{T}\to\infty$, where $u_{\mathbf{T}}^{*}$ and $u_{\mathbf{T}}^{*'}$ are defined in Lemma 3.4. Using the homogeneity of $\{\eta(\mathbf{t}),\mathbf{t}\ge\mathbf{0}\}$,

$$\begin{aligned}
&\prod_{i_1=1}^{n_1}\prod_{i_2=1}^{n_2}P\left\{\max_{\mathbf{t}\in\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap\mathbf{O_i}}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\\
&=\left(P\left\{\max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\right)^{n_1n_2}\\
&=\exp\left(n_1n_2\log\left(P\left\{\max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\right)\right)\\
&=\exp\left(-n_1n_2\left(1-P\left\{\max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\right)+R_{\mathbf{n}}\right).
\end{aligned}$$

Since

$$P_{\mathbf{n}}=:P\left\{\max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\to 1,$$

as $\mathbf{T}\to\infty$, we get that the remainder $R_{\mathbf{n}}$ can be estimated as $R_{\mathbf{n}}=o(n_1n_2(1-P_{\mathbf{n}}))$. Using Lemma A2, (10) and (11), we get that

$$n_1n_2\left(1-P\left\{\max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*},\max_{\mathbf{t}\in\Re(p_1)\times\Re(p_2)\cap[0,T_1^a]\times[0,T_2^a]}\eta(\mathbf{t})\le u_{\mathbf{T}}^{*'}\right\}\right)$$

$$
\begin{aligned}
&\sim n_1 n_2 T_1^a T_2^a T_1^{-1} T_2^{-1} \left(e^{-x-r+\sqrt{2r}z} + e^{-y-r+\sqrt{2r}z}\right) \\
&\sim e^{-x-r+\sqrt{2r}z} + e^{-y-r+\sqrt{2r}z},
\end{aligned}
$$

which combined with the dominated convergence theorem completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. The proof of the first assertion of Theorem 2.2 can be found in Subsection 4.2. Next, we give the proof of the second assertion. In view of Lemmas 3.1-3.4 in order to establish the proof we need to show

$$
\left| \int_{-\infty}^{+\infty} \prod_{i_1=1}^{n_1} \prod_{i_2=1}^{n_2} P\left\{ \max_{\mathbf{t}\in\mathbf{O_i}} \eta(\mathbf{t}) \le u_{\mathbf{T}}^*, \max_{t\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap\mathbf{O_i}} \eta(\mathbf{t}) \le u_{\mathbf{T}}^{*'} \right\} \phi(z)dz \right.
$$
$$
\left. - \int_{-\infty}^{+\infty} \exp\left(-\left(e^{-x-r+\sqrt{2r}z} + e^{-y-r+\sqrt{2r}z} - H_{a,\alpha}^{\log H_{a,\alpha}+x, \log H_\alpha + y} e^{-r+\sqrt{2r}z}\right)\right) \phi(z)dz \right| \to 0
$$

as $\mathbf{T}\to\infty$, where $u_{\mathbf{T}}^*$ and $u_{\mathbf{T}}^{*'}$ are defined in Lemma 3.4. Similar to the proof of Theorem 2.1, using Lemma A3, we get

$$
\begin{aligned}
& n_1 n_2 \left(1 - P\left\{ \max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) \le u_{\mathbf{T}}^*, \max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) \le u_{\mathbf{T}}^{*'} \right\}\right) \\
&= n_1 n_2 P\left\{ \max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^* \right\} + n_1 n_2 P\left\{ \max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^{*'} \right\} \\
&\quad - n_1 n_2 P\left\{ \max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^*, \max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^{*'} \right\} \\
&\sim n_1 n_2 T_1^a T_2^a T_1^{-1} T_2^{-1} \left(e^{-x-r+\sqrt{2r}z} + e^{-y-r+\sqrt{2r}z}\right) \\
&\quad - n_1 n_2 P\left\{ \max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^*, \max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^{*'} \right\},
\end{aligned}
$$

as $\mathbf{T}\to\infty$. To transform the last term, using (10) and (11), we get

$$
\begin{aligned}
u_{\mathbf{T}}^* &= \frac{x + r - \sqrt{2r}z}{a_{\mathbf{T}}} + b_{\mathbf{T}} + o(a_{\mathbf{T}}^{-1}) \\
&= u_{\mathbf{T}}^{*'} + b_{\mathbf{T}} - b_{a,\mathbf{T}} + (x-y)/a_{\mathbf{T}} \\
&= u_{\mathbf{T}}^{*'} + \frac{\log(\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}) - \log(\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2}) + x - y}{u_{\mathbf{T}}^{*'}} + O\left((\log\log(T_1T_2))^2 (\log T_1T_2)^{-3/2}\right).
\end{aligned}
$$

Observing that $u_{\mathbf{T}}^{*'} \sim (2\log T_1T_2)^{1/2}$, we see that the reminder $O(\cdot)$ plays a negligible role. Therefore, by (21) in Appendix A

$$
\begin{aligned}
& n_1 n_2 P\left\{ \max_{\mathbf{t}\in[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^*, \max_{\mathbf{t}\in\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)\cap[0,T_1^a]\times[0,T_2^a]} \eta(\mathbf{t}) > u_{\mathbf{T}}^{*'} \right\} \\
&= n_1 n_2 T_1^a T_2^a \mathcal{H}_{\mathbf{a},\alpha_1,\alpha_2}^{0,Z_{x,y}} (u_T^{*'})^{2/\alpha_1 + 2/\alpha_2} \Psi(u_T^{*'})(1+o(1)) \\
&= n_1 n_2 T_1^a T_2^a (T_1T_2)^{-1} \mathcal{H}_{\mathbf{a},\alpha_1,\alpha_2}^{0,Z_{x,y}} (\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2})^{-1} e^{-y-r+\sqrt{2r}z}(1+o(1)),
\end{aligned}
$$

where $Z_{x,y} = \log(\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}) - \log(\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2}) + x - y$. Next, changing the variables in the definition of $\mathcal{H}_{\mathbf{a},\alpha_1,\alpha_2}^{x,y}$ we get that $\mathcal{H}_{\mathbf{a},\alpha_1,\alpha_2}^{0,Z_{x,y}} (\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2})^{-1} e^{-y} = \mathcal{H}_{\mathbf{a},\alpha_1,\alpha_2}^{\log(\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2})+x, \log(\mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2})+y}$. This and the dominated convergence theorem conclude the proof of Theorem 2.2. □

Proof of Theorem 2.3. In view of Lemma A4 we have

$$
\begin{aligned}
& \left| P\left\{ a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le x, a_{\mathbf{T}}(M_{\mathbf{T}}^{\mathbf{p}} - b_{\mathbf{T}}) \le y \right\} - P\left\{ a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le x, a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le y \right\} \right| \\
&\le \left| P\left\{ a_{\mathbf{T}}(M_{\mathbf{T}}^{\mathbf{p}} - b_{\mathbf{T}}) \le y \right\} - P\left\{ a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le y \right\} \right| \to 0, \quad \mathbf{T}\to\infty.
\end{aligned}
$$

Next, applying Corollary 2.1, we get

$$
\begin{aligned}
P\left\{ a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le x, a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le y \right\} &= P\{a_{\mathbf{T}}(M_{\mathbf{T}} - b_{\mathbf{T}}) \le \min(x,y)\} \\
&\to \int_{-\infty}^{+\infty} \exp\left(-e^{-\min(x,y) - r + \sqrt{2r}z}\right) \phi(z)dz, \quad \mathbf{T}\to\infty,
\end{aligned}
$$

hence the proof is complete. □

4 Appendix A

In this section, we give some auxiliary results, which extend Lemmas 1-4 of Piterbarg (2004) from stationary Gaussian processes to Gaussian random fields. The ideas of the proofs are very close to that of the above mentioned lemmas. In the following subsections, we suppose that Assumptions **A1** and **A2** hold.

4.1 Sparse grid

In this subsection, suppose $\Re(p_i),\ i=1,2$ are sparse grids. We will use the notations $u=\sqrt{2\log T_1T_2}$, so that $p_i=p_i(u)=l_i(u)u^{-2/\alpha_i}$, $i=1,2$, where $l_i(u)\to\infty$ as $u\to\infty$, with $p_i(u)\le p_0$ for some positive p_0, in particular, $p_i(u)=p_0$. Let $\mathbf{L_p}=[-p_1,p_1]\times[-p_2,p_2]$. First we consider the following probability

$$P(u,x)=P\left(X(\mathbf{0})>u,\max_{\mathbf{t}\in\mathbf{L_p}}X(\mathbf{t})>u+\frac{(\frac{2}{\alpha_1}+\frac{2}{\alpha_2})\log u+\log(p_1p_2)+x}{u}\right),$$

where x is varies in a closed interval, say, $x\in[-A,A]$ with $A<\infty$. For simplicity, we denote

$$v:=\sqrt{(\frac{2}{\alpha_1}+\frac{2}{\alpha_2})\log u+\log(p_1p_2)}.$$

By (4) (see also Theorem 7.1 of Piterbarg (1996)), we have

$$\begin{aligned}P\left(\max_{\mathbf{t}\in\mathbf{L_p}}X(\mathbf{t})>u+\frac{v^2+x}{u}\right)&=4p_1p_2\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}\left(u+\frac{v^2+x}{u}\right)^{2/\alpha_1+2/\alpha_2}\Psi\left(u+\frac{v^2+x}{u}\right)(1+o(1))\\&=4\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}e^{-x}\Psi(u)(1+o(1))\end{aligned}\tag{14}$$

as $u\to\infty$.

Lemma A1. *We have $P(u,x)=o(\Psi(u))$ as $u\to\infty$.*

Proof: Write $w=\left((\frac{2}{\alpha_1}+\frac{2}{\alpha_2})\log u+\log(p_1p_2)+x\right)/u$. We have,

$$P(u,x)\le P\left(\max_{\mathbf{t}\in\mathbf{L_p}}(X(\mathbf{0})+X(\mathbf{t}))>2u+w\right).$$

Let p_1',p_2' be so small that $1-r(\mathbf{t})\le 2(|t_1|^{\alpha_1}+|t_2|^{\alpha_2})$ for all $\mathbf{t}\in\mathbf{I_{p'}}=[-p_1',p_1']\times[-p_2',p_2']$. If $\mathbf{L_p}\cap\mathbf{L_{p'}}\ne\emptyset$, write

$$\begin{aligned}P\left(\max_{\mathbf{t}\in\mathbf{L_p}}(X(\mathbf{0})+X(\mathbf{t}))>2u+w\right)&\le P\left(\max_{\mathbf{t}\in\mathbf{L_{p'}}}(X(\mathbf{0})+X(\mathbf{t}))>2u+w\right)\\&+P\left(\max_{\mathbf{t}\in\mathbf{L_p}\setminus\mathbf{L_{p'}}}(X(\mathbf{0})+X(\mathbf{t}))>2u+w\right).\end{aligned}\tag{15}$$

The variance of the field $X(\mathbf{0})+X(\mathbf{t})$, $\mathbf{t}\in\mathbf{L_p}\setminus\mathbf{L_{p'}}$, is less than $4-\varepsilon$ for sufficiently small $\varepsilon>0$ and that

$$E[(X(\mathbf{0})+X(\mathbf{t}))-(X(\mathbf{0})+X(\mathbf{s}))]^2=2(|t_1-s_1|^{\alpha_1}+|t_2-s_2|^{\alpha_1})(1+o(1))$$

as $\mathbf{t}-\mathbf{s}\to\mathbf{0}$, so by Theorem 8.1 of Piterbarg (1996), for all sufficiently large u and some positive $\varepsilon'<\varepsilon$,

$$\begin{aligned}P\left(\max_{\mathbf{t}\in\mathbf{L_p}\setminus\mathbf{L_{p'}}}(X(\mathbf{0})+X(\mathbf{t}))>2u+w\right)&\le Cp_1p_2(2u+w)^{2/\alpha_1+2/\alpha_2}\Psi\left(\frac{2u+w}{\sqrt{4-\varepsilon}}\right)\\&\le C(u)^{2/\alpha_1+2/\alpha_2-1}\exp\left(-\frac{u^2}{2-\varepsilon'/2}\right)\\&=o(\Psi(u)),\end{aligned}$$

as $u\to\infty$.

We will apply Theorem 8.2 of Piterbarg (1996), for the first probability in the right-hand part of (15). To this end, by some simple calculations, we get for the correlation function of the field $X(\mathbf{0})+X(\mathbf{t})$, $\mathbf{t}\in\mathbf{L_{p'}}$

$$1-\frac{E(X(\mathbf{0})+X(\mathbf{t}))(X(\mathbf{0})+X(\mathbf{s}))}{\sqrt{E(X(\mathbf{0})+X(\mathbf{t}))^2E(X(\mathbf{0})+X(\mathbf{s}))^2}}\le\frac{1-r(\mathbf{t}-\mathbf{s})}{2\sqrt{1+r(\mathbf{t})}\sqrt{1+r(\mathbf{s})}}$$

$$
\begin{aligned}
&\leq \frac{|t_1 - s_1|^{\alpha_1} + |t_2 - s_2|^{\alpha_2}}{2(2 - \delta_1'^{\alpha_1} - \delta_2'^{\alpha_2})} \\
&\leq 1 - \exp(-|t_1 - s_1|^{\alpha_1} - |t_2 - s_2|^{\alpha_2}),
\end{aligned}
$$

where we assume an additionally that $p_1'^{\alpha_1} + p_2'^{\alpha_2} \leq 3/2$. For the variance of the field $X(\mathbf{0}) + X(\mathbf{t})$, $\mathbf{t} \in \mathbf{L}_{\mathbf{p}'}$ we have

$$
Var(X(\mathbf{0}) + X(\mathbf{t})) = 2 + 2r(\mathbf{t}) = 4 - 2(|t_1|^{\alpha_1} + |t_2|^{\alpha_2})(1 + o(1))
$$

as $\mathbf{t} \to 0$, and the point $\mathbf{t} = \mathbf{0}$ is the unique point of maximum of variance of the field $X(\mathbf{0}) + X(\mathbf{t})$. By Slepian's inequality

$$
\begin{aligned}
&P\left(\max_{\mathbf{t}\in\mathbf{L}_{\mathbf{p}'}} (X(\mathbf{0}) + X(\mathbf{t})) > 2u + w\right) \\
&= P\left(\max_{\mathbf{t}\in\mathbf{L}_{\mathbf{p}'}} \frac{X(\mathbf{0}) + X(\mathbf{t})}{\sqrt{E(X(\mathbf{0}) + X(\mathbf{t}))^2}} \sqrt{E(X(\mathbf{0}) + X(\mathbf{t}))^2} > 2u + w\right) \\
&\leq P\left(\max_{\mathbf{t}\in\mathbf{L}_{\mathbf{p}'}} Y(\mathbf{t}) \sqrt{E(X(\mathbf{0}) + X(\mathbf{t}))^2} > 2u + w\right),
\end{aligned}
$$

where $Y(\mathbf{t})$ is a Gaussian zero mean homogeneous field with covariance function $\exp(-[|t_1|^{\alpha_1} + |t_2|^{\alpha_2}])$, and thus the conditions of Theorem 8.2 of Piterbarg (1996), for the case (ii) holds. By this theorem, for some constants C, C',

$$
\begin{aligned}
&P\left(\max_{\mathbf{t}\in\mathbf{L}_{\mathbf{p}'}} Y(\mathbf{t}) \sqrt{E(X(\mathbf{0}) + X(\mathbf{t}))^2} > 2u + w\right) \\
&= C\Psi(u + w/2)(1 + o(1)) \\
&= C'u^{-1}\exp(-u^2/2 - uw)(1 + o(1)) \\
&= C'\Psi(u)\exp\left(-\frac{1}{2}((\frac{2}{\alpha_1} + \frac{2}{\alpha_2})\log u + \log(p_1p_2) + x)\right)(1 + o(1)) \\
&= C'\Psi(u)e^{-1/2x}\left(u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}}p_1p_2\right)^{-1/2}(1 + o(1)) \\
&= C'\Psi(u)e^{-1/2x}\left((2\log T_1T_2)^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}p_1p_2\right)^{-1/2}(1 + o(1)).
\end{aligned}
$$

Since $(2\log T_1T_2)^{1/\alpha_i}p_i \to \infty$ for sparse grids $\mathfrak{R}(p_i)$, we get the assertion of the lemma. $\square$

Now we consider the probability

$$
P_{\mathbf{S}}(u, x) = P\left(\max_{\mathbf{t}\in\mathbf{I}_{\mathbf{S}}\cap\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)} X(\mathbf{t}) > u, \max_{\mathbf{t}\in\mathbf{I}_{\mathbf{S}}} X(\mathbf{t}) > u + \frac{v^2 + x}{u}\right)
$$

when we will allow S_1S_2 tends to infinity with u but not too fast. Define

$$
\delta(\varepsilon) = \inf_{\max\{|t_1|,|t_2|\}\geq\varepsilon} (1 - r(\mathbf{t})).
$$

Note that $\delta(\varepsilon)$ is positive for all positive ε.

Lemma A2. *Let $S_i = S_i(u) \geq 2p_i$ for all u, $i = 1, 2$ and $S_1S_2u^{2/\alpha_1+2/\alpha_2} = o(\exp(u^2\delta(\varepsilon)/8))$ as $u \to \infty$. Then there exists an $\varepsilon > 0$ such that*

$$
P\left(\max_{\mathbf{t}\in\mathbf{I}_{\mathbf{S}}\cap\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)} X(\mathbf{t}) > u\right) \sim S_1S_2p_1^{-1}p_2^{-1}\Psi(u), \tag{16}
$$

$$
P\left(\max_{\mathbf{t}\in\mathbf{I}_{\mathbf{S}}} X(\mathbf{t}) > u + \frac{v^2 + x}{u}\right) \sim S_1S_2p_1^{-1}p_2^{-1}e^{-x}\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}\Psi(u), \tag{17}
$$

as $u \to \infty$ and

$$
P_{\mathbf{S}}(u, x) = o\left(P\left(\max_{\mathbf{t}\in\mathbf{I}_{\mathbf{S}}\cap\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)} X(\mathbf{t}) > u\right) + P\left(\max_{\mathbf{t}\in\mathbf{I}_{\mathbf{S}}} X(\mathbf{t}) > u + \frac{v^2 + x}{u}\right)\right)
$$

as $u \to \infty$ so that

$$
\begin{aligned}
&1 - P\left(\max_{\mathbf{t}\in\mathbf{I_S}\cap\Re(p_1)\times\Re(p_2)} X(\mathbf{t}) \le u, \max_{\mathbf{t}\in\mathbf{I_S}} X(\mathbf{t}) \le u + \frac{v^2+x}{u}\right) \\
&\sim P\left(\max_{\mathbf{t}\in\mathbf{I_S}\cap\Re(p_1)\times\Re(p_2)} X(\mathbf{t}) > u\right) + P\left(\max_{\mathbf{t}\in\mathbf{I_S}} X(\mathbf{t}) > u + \frac{v^2+x}{u}\right) \\
&\sim S_1S_2p_1^{-1}p_2^{-1}\Psi(u)(1 + e^{-x}\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2})
\end{aligned}
\tag{18}
$$

as $u \to \infty$.

Proof: The relation (17) is in fact a special case of Theorem 7.2 of Piterbarg (1996) and relation (16) can be proved by the same way. Now we prove that for a sparse grid the double probability $P_{\mathbf{S}}(u,x)$ tends to zero faster than right-hand part of (17). Let $\mathbf{J_l} = [(l_1-1)p_1, (l_1+1)p_1] \times [(l_2-1)p_2, (l_2+1)p_2]$, where $l_i = 0, 1, 2, \cdots, [S_i/p_i]$, $i = 1, 2$. We have

$$
\begin{aligned}
P_{\mathbf{S}}(u,x) &\le \sum_{k_1,l_1=0}^{[S_1/p_1]}\sum_{k_2,l_2=0}^{[S_2/p_2]} P\left(X(k_1p_1,k_2p_2) > u, \max_{\mathbf{t}\in\mathbf{J_l}} X(\mathbf{t}) > u + \frac{v^2+x}{u}\right) =: \sum_{k_1,l_1=0}^{[S_1/p_1]}\sum_{k_2,l_2=0}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}} \\
&= \sum_{k_1,l_1=0,|k_1-l_1|\le 1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|\le 1}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}} + \sum_{k_1,l_1=0,|k_1-l_1|\le 1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|> 1}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}} \\
&+ \sum_{k_1,l_1=0,|k_1-l_1|> 1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|\le 1}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}} + \sum_{k_1,l_1=0,|k_1-l_1|> 1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|> 1}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}}.
\end{aligned}
\tag{19}
$$

The members of the first term on the right-hand side of (19) can be estimated by Lemma A1, so that

$$
\sum_{k_1,l_1=0,|k_1-l_1|\le 1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|\le 1}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}} = S_1S_2p_1^{-1}p_2^{-1}o(\Psi(u))
\tag{20}
$$

as $u \to \infty$. Let $\mathbf{m} = (m_1, m_2)$ with $m_i = 0, 1, 2, \cdots, [S_i/p_i]$, $i = 1, 2$. Now consider the probability $P_{\mathbf{k},\mathbf{k}+\mathbf{m}} = P_{\mathbf{0},\mathbf{m}}$ for $\max\{m_1, m_2\} > 1$. We have, using Theorem 8.1 of Piterbarg (1996),

$$
\begin{aligned}
P_{\mathbf{0},\mathbf{m}} &\le P\left(\max_{\mathbf{t}\in\mathbf{J_m}}(X(\mathbf{0}) + X(\mathbf{t})) > 2u + \frac{v^2+x}{u}\right) \\
&\le Cp_1p_2u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1}\exp\left(-\frac{(2u + (v^2+x)/u)^2}{2\max_{\mathbf{t}\in\mathbf{J_m}}(2+2r(\mathbf{t}))}\right) \\
&\le Cp_1p_2u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1}\exp\left(-\frac{u^2+v^2}{2(1-\frac{1}{2}\min_{\mathbf{t}\in\mathbf{J_m}}(1-r(\mathbf{t}))}\right) \\
&\le Cp_1p_2u^{\frac{2}{\alpha_1}+\frac{2}{\alpha_2}-1}\exp\left(-\frac{1}{2}(u^2+v^2)(1+\frac{1}{2}\min_{\mathbf{t}\in\mathbf{J_m}}(1-r(\mathbf{t}))\right) \\
&\le Cp_1^{1/2}p_2^{1/2}u^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}-1}\exp\left(-\frac{1}{2}u^2\right)\exp\left(-\frac{1}{4}u^2\min_{\mathbf{t}\in\mathbf{J_m}}(1-r(\mathbf{t}))\right) \\
&\le Cp_1^{1/2}p_2^{1/2}u^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\Psi(u)\exp\left(-\frac{1}{4}u^2\min_{\mathbf{t}\in\mathbf{J_m}}(1-r(\mathbf{t}))\right).
\end{aligned}
$$

Let ε be such that $1 - r(\mathbf{t}) \ge \frac{1}{2}(|t_1|^{\alpha_1} + |t_2|^{\alpha_2})$ for all $\mathbf{t} \in (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$. Then

$$
P_{\mathbf{0},\mathbf{m}} \le Cp_1^{1/2}p_2^{1/2}u^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\Psi(u)\exp\left(-\frac{1}{8}u^2\delta(\varepsilon)\right)
$$

for $\max\{|(m_1-1)p_1|, |(m_2-1)p_2|\} > \varepsilon$ and

$$
P_{\mathbf{0},\mathbf{m}} \le Cp_1^{1/2}p_2^{1/2}u^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\Psi(u)\exp\left(-\frac{1}{8}u^2[|(m_1-1)p_1|^{\alpha_1} + |(m_2-1)p_2|^{\alpha_2}]\right)
$$

for $\max\{|(m_1-1)p_1|, |(m_2-1)p_2|\} \le \varepsilon$. Thus, letting $\mathbf{i} = \mathbf{l} - \mathbf{k}$, for the second sum we have

$$
\sum_{k_1,l_1=0,|k_1-l_1|\le 1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|> 1}^{[S_2/p_2]} P_{\mathbf{k},\mathbf{l}} \le 4S_1S_2p_1^{-1}p_2^{-1}\sum_{i_1=0}^{1}\sum_{i_2=2}^{[S_2/p_2]} p_{\mathbf{0},\mathbf{i}}
$$

$$
\begin{aligned}
&\leq CS_1S_2p_1^{-1/2}p_2^{-1/2}u^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\Psi(u)\Big\{S_2p_2^{-1}\exp\left(-\frac{1}{8}u^2\delta(\varepsilon)\right)\\
&\quad+\sum_{i_1-1=-1}^{0}\sum_{i_2-1=1}^{\varepsilon/p_2}\exp\left(-\frac{1}{8}u^2[|(i_1-1)p_1|^{\alpha_1}+|(i_2-1)p_2|^{\alpha_2}]\right)\Big\}\\
&\leq CS_1S_2p_1^{-1}p_2^{-1}\Psi(u)o(1)
\end{aligned}
$$

as $u\to\infty$. Similarly,

$$
\sum_{k_1,l_1=0,|k_1-l_1|>1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|\leq 1}^{[S_2/p_2]}P_{\mathbf{k},\mathbf{l}}\leq CS_1S_2p_1^{-1}p_2^{-1}\Psi(u)o(1)
$$

as $u\to\infty$. For the fourth sum, we have

$$
\begin{aligned}
\sum_{k_1,l_1=0,|k_1-l_1|>1}^{[S_1/p_1]}\sum_{k_2,l_2=0,|k_2-l_2|>1}^{[S_2/p_2]}P_{\mathbf{k},\mathbf{l}} &\leq 4S_1S_2p_1^{-1}p_2^{-1}\sum_{i_1=2}^{[S_1/\delta_1]}\sum_{i_2=2}^{[S_2/p_2]}P_{\mathbf{0},\mathbf{i}}\\
&\leq CS_1S_2p_1^{-1/2}p_2^{-1/2}u^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\Psi(u)\Big\{S_1p_1^{-1}S_2p_2^{-1}\exp\left(-\frac{1}{8}u^2\delta(\varepsilon)\right)\\
&\quad+\sum_{i_1-1=1}^{\varepsilon/p_1}\sum_{i_2-1=1}^{\varepsilon/p_2}\exp\left(-\frac{1}{8}u^2[|(i_1-1)p_1|^{\alpha_1}+|(i_2-1)p_2|^{\alpha_2}]\right)\Big\}\\
&\leq CS_1S_2p_1^{-1}p_2^{-1}\Psi(u)o(1),
\end{aligned}
$$

as $u\to\infty$. Now we can easily prove the relation (17). We have for all $\mathbf{k}$ and $\mathbf{l}$

$$
P\left(X(k_1p_1,k_2p_2)>u,X(l_1p_1,l_2p_2)>u\right)\leq P_{\mathbf{k},\mathbf{l}},
$$

hence

$$
\sum_{k_1,l_1=0,k_1\neq l_1}^{[S_1/p_1]}\sum_{k_2,l_2=0,k_2\neq l_2}^{[S_2/p_2]}P\left(X(k_1p_1,k_2p_2)>u,X(l_1p_1,l_2p_2)>u\right)\leq\sum_{k_1,l_1=0,k_1\neq l_1}^{[S_1/p_1]}\sum_{k_2,l_2=0,k_2\neq l_2}^{[S_2/p_2]}P_{\mathbf{k},\mathbf{l}},
$$

from which it follows that double sum in the above left-hand side tends to zero faster than $S_1S_2p_1^{-1}p_2^{-1}\Psi(u)$ as $u\to\infty$. Thus, both the assertions of Lemma A2 are proved. □

4.2 Pickands grid

Lat $\mathbf{a}=(a_1,a_2)>(0,0)$. In this subsection suppose that $\mathfrak{R}(p_i)$, $i=1,2$ are Pickands grids, ie., $\mathfrak{R}(p_i)=\{a_iku^{-2/\alpha_i},k\in\mathbb{N}\}$. We will evaluate the asymptotic behavior of the probability

$$
P'_{\mathbf{S}}(u,x)=P\left(\max_{\mathbf{t}\in\mathbf{I_S}\cap\mathfrak{R}(p_1)\times\mathfrak{R}(p_2)}X(\mathbf{t})>u,\max_{\mathbf{t}\in\mathbf{I_S}}X(\mathbf{t})>u+\frac{x}{u}\right).
$$

As in the previous subsection, we begin with a short interval. Let $\lambda_i>a_i$. Then it can be proved quite similar to the proof of Lemma 6.1 of Piterbarg (1996), that

$$
P'_{(\lambda_1u^{-2/\alpha_1},\lambda_2u^{-2/\alpha_2})}(u,x)\sim\mathcal{H}^{0,x}_{\mathbf{a},\alpha_1,\alpha_2}\Psi(u)
$$

as $u\to\infty$, where

$$
H^{0,x}_{\mathbf{d},\alpha_1,\alpha_2}(\lambda_1,\lambda_2)=\int_{-\infty}^{+\infty}e^sP\left(\max_{(k_1d_1,k_2d_2)\in[0,\lambda_1]\times[0,\lambda_2]}\sqrt{2}\chi(k_1d_1,k_2d_2)>s,\max_{(t_1,t_2)\in[0,\lambda_1]\times[0,\lambda_2]}\sqrt{2}\chi(t_1,t_2)>s+x\right)ds.
$$

It also can be proved in a similar way as for Lemma 6.1 and Theorem 7.2 of Piterbarg (1996) that

$$
\mathcal{H}^{0,x}_{\mathbf{a},\alpha_1,\alpha_2}:=\lim_{\substack{\lambda_1\to\infty\\ \lambda_2\to\infty}}\mathcal{H}^{0,x}_{\mathbf{a},\alpha_1,\alpha_2}(\lambda_1,\lambda_2)/(\lambda_1\lambda_2)\in(0,\infty)
$$

and that there exsits $\kappa \in (0,1/2)$ such that for any $S_i = S_i(u)$ with $S_1 S_2 u^{-(2/\alpha_1+2/\alpha_2)} \to \infty$ and $S_1 S_2 = O(\exp(\kappa u^2))$ as $u \to \infty$ with

$$P'_{\mathbf{S}}(u,x) \sim S_1 S_2 \mathcal{H}^{0,x}_{\mathbf{a},\alpha_1,\alpha_2} u^{2/\alpha_1+2/\alpha_2} \Psi(u) \tag{21}$$

as $u \to \infty$, respectively. From here we have for Pickands grids,

Lemma A3. *For any* $\mathbf{a} = (a_1, a_2)$ *and* $\Re(p_i) = \{a_i k u^{-2/\alpha_i}, k \in \mathbb{N}\}$,

$$\begin{aligned}
&1 - P\left(\max_{\mathbf{t}\in \mathbf{I_S}\cap\Re(p_1)\times\Re(p_2)} X(\mathbf{t}) \le u, \max_{\mathbf{t}\in\mathbf{I_S}} X(\mathbf{t}) \le u + \frac{v^2+x}{u}\right)\\
&= P\left(\max_{\mathbf{t}\in \mathbf{I_S}\cap\Re(p_1)\times\Re(p_2)} X(\mathbf{t}) > u\right) + P\left(\max_{\mathbf{t}\in\mathbf{I_S}} X(\mathbf{t}) > u + \frac{v^2+x}{u}\right)\\
&\quad - P\left(\max_{\mathbf{t}\in \mathbf{I_S}\cap\Re(p_1)\times\Re(p_2)} X(\mathbf{t}) > u, \max_{\mathbf{t}\in\mathbf{I_S}} X(\mathbf{t}) > u + \frac{x}{u}\right)\\
&\sim S_1 S_2 \mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2}\left(u + \frac{v^2+x}{u}\right)^{2/\alpha_1+2/\alpha_2} \Psi\left(u + \frac{v^2+x}{u}\right) + S_1 S_2 \mathcal{H}_{a_1,\alpha_1}\mathcal{H}_{a_2,\alpha_2} u^{2/\alpha_1+2/\alpha_2}\Psi(u)\\
&\quad - S_1 S_2 \mathcal{H}^{0,x}_{\mathbf{a},\alpha_1,\alpha_2} u^{2/\alpha_1+2/\alpha_2}\Psi(u)
\end{aligned} \tag{22}$$

as $u \to \infty$.

4.3 Dense grid

In this subsection, we state a lemma for the dense grid case which is important for our proofs.

Lemma A4. *Let* $S_i = S_i(u)$ *with* $S_1 S_2 u^{-(2/\alpha_1+2/\alpha_2)} \to \infty$ *and* $S_1 S_2 = O(\exp(\kappa u^2))$ *with* $\kappa \in (0,1/2]$ *as* $u \to \infty$. *For any* $\mathbf{a} = (a_1, a_2)$ *and* $\Re(p_i) = \{a_i k u^{-2/\alpha_i}, k \in \mathbb{N}\}$, *we have*

$$P\left(\max_{\mathbf{t}\in \mathbf{I_S}\cap\Re(p_1)\times\Re(p_2)} X(\mathbf{t}) \le u\right) - P\left(\max_{\mathbf{t}\in\mathbf{I_S}} X(\mathbf{t}) \le u\right) = g(a_1,a_2)\mathcal{H}_{\alpha_1}\mathcal{H}_{\alpha_2} S_1 S_2 u^{2/\alpha_1+2/\alpha_2}\Psi(u), \tag{23}$$

where $g(a_1, a_2) \to 0$ *as* $\mathbf{a} \to \mathbf{0}$.

Proof: Lemma 1 of Dębicki et al. (2014) shows that (23) holds for some fixed $S_i > 0$. By the homogeneity of $X(\mathbf{t})$, it is easy to extend (23) to the case $S_1 S_2 = O(\exp(\kappa u^2))$ with $\kappa \in (0,1/2]$, see eg. the proof of Lemma 12.3.2 of Leadbetter (1983) for more details. □

5 Appendix B

In this section, we give three technical lemmas which are used for the proof of Lemma 3.1. Recall that $u_{\mathbf{T}} = b_{\mathbf{T}} + x/a_{\mathbf{T}}$, $u'_{\mathbf{T}} = b'_{\mathbf{T}} + y/a_{\mathbf{T}}$, where $b'_{\mathbf{T}} = b^{\mathbf{p}}_{\mathbf{T}}$ for sparse grids and $b'_{\mathbf{T}} = b_{\mathbf{a},\mathbf{T}}$ for Pickands grids, and $r^{(h)}(\mathbf{kq},\mathbf{lq}) = h r(\mathbf{kq},\mathbf{lq}) + (1-h)\varrho(\mathbf{kq},\mathbf{lq})$ with $h \in [0,1]$. Let

$$\varpi(\mathbf{t},\mathbf{s}) = \max\{|r(\mathbf{t},\mathbf{s})|, |\varrho(\mathbf{t},\mathbf{s})|\}$$

and

$$\vartheta(\mathbf{z}) = \sup_{\substack{\mathbf{0}\le\mathbf{s},\mathbf{t}\le\mathbf{T},\\ \{|s_1-t_1|>z_1\}\cup\{|s_2-t_2|>z_2\}}} \{\varpi(\mathbf{t},\mathbf{s})\}.$$

It is easy to see from Assumptions **A1** and **A2** that for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$

$$\vartheta(\varepsilon_1, \varepsilon_2) < 1$$

for all sufficiently large $\mathbf{T}$. Further, let a, b be such that

$$0 < b < a < \big(1 - \vartheta(\varepsilon,\varepsilon)\big)/\big(1 + \vartheta(\varepsilon,\varepsilon)\big) < 1$$

for all sufficiently large $\mathbf{T}$ and for some $\varepsilon > 0$ which will be chosen in the blow.

Lemma B1. *Under the conditions of Lemma 3.3, we have*

$$\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j}\\ \mathbf{kq}\neq\mathbf{lq},1\le\mathbf{i},\mathbf{j}\le\mathbf{n}}} |r(\mathbf{kq},\mathbf{lq})-\varrho(\mathbf{kq},\mathbf{lq})|\int_0^1 \frac{1}{\sqrt{1-r^{(h)}(\mathbf{kq},\mathbf{lq})}}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+r^{(h)}(\mathbf{kq},\mathbf{lq})}\right)dh\to 0 \tag{24}$$

as $\mathbf{T}\to\infty$.

Proof: Recall that $\Re(q_i)$, $i=1,2$ are Pickands grids. First, we consider the case that $\mathbf{kq},\mathbf{lq}$ in the same interval $\mathbf{O_i}$. Split the sum (24) into two parts as

$$\sum_{\substack{\mathbf{kq},\mathbf{lq}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lq},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1q_1-k_1q_1|,|l_2q_2-k_2q_2|\}\le\varepsilon}} + \sum_{\substack{\mathbf{kq},\mathbf{lq}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lq},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1q_1-k_1q_1|,|l_2q_2-k_2q_2|\}>\varepsilon}} =: J_{\mathbf{T},1}+J_{\mathbf{T},2}. \tag{25}$$

We deal with $J_{\mathbf{T},1}$ and note that in this case, by the definition of the field $\xi_{\mathbf{T}}(\mathbf{t})$, we have $\varrho(\mathbf{kq},\mathbf{lq})-r(\mathbf{kq},\mathbf{lq}) = \rho(\mathbf{T})(1-r(\mathbf{kq},\mathbf{lq}))$. By Assumption **A1** we can choose small enough $\varepsilon>0$ such that $\varrho(\mathbf{kq},\mathbf{lq})=r(\mathbf{kq},\mathbf{lq})+(1-r(\mathbf{kq},\mathbf{lq}))\rho(\mathbf{T})\sim r(\mathbf{kq},\mathbf{lq})$ for sufficiently large $\mathbf{T}$ and $\max\{|l_1q_1-k_1q_1|,|l_2q_2-k_2q_2|\}\le\varepsilon$. It follows from Assumption **A1** again that for all $|t_i|\le\varepsilon<2^{-1/\alpha_i}$,

$$\frac{1}{2}(|t_1|^{\alpha_1}+|t_2|^{\alpha_2})\le 1-r(\mathbf{t})\le 2(|t_1|^{\alpha_1}+|t_2|^{\alpha_2}) \tag{26}$$

and the definition of $u_{\mathbf{T}}$ implies

$$u_{\mathbf{T}}^2 = 2\log T_1T_2-\log\log T_1T_2+(\frac{2}{\alpha_1}+\frac{2}{\alpha_2})\log\log T_1T_2+O(1). \tag{27}$$

Consequently, since further $q_i=\gamma_i(\log T_1T_2)^{-1/\alpha_i}$ we obtain

$$\begin{aligned}
J_{\mathbf{T},1} &\le C\sum_{\substack{\mathbf{kq},\mathbf{lq}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lq},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1q_1-k_1q_1|,|l_2q_2-k_2q_2|\}\le\varepsilon}} |r(\mathbf{kq},\mathbf{lq})-\varrho(\mathbf{kq},\mathbf{lq})|\frac{1}{\sqrt{1-r(\mathbf{kq},\mathbf{lq})}}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+r(\mathbf{kq},\mathbf{lq})}\right)\\
&\le C\frac{T_1}{q_1}\frac{T_2}{q_2}\rho(\mathbf{T})\sum_{0<k_1q_1\le\varepsilon,0<k_2q_2\le\varepsilon}|1-r(\mathbf{kq})|\frac{1}{\sqrt{1-r(\mathbf{kq})}}\exp\left(-\frac{u_{\mathbf{T}}^2}{2}\right)\exp\left(-\frac{(1-r(\mathbf{kq}))u_{\mathbf{T}}^2}{2(1+r(\mathbf{kq}))}\right)\\
&\le C\frac{T_1}{q_1}\frac{T_2}{q_2}\rho(\mathbf{T})T_1^{-1}T_2^{-1}(\log T_1T_2)^{1/2-1/\alpha_1-1/\alpha_2}\sum_{0<k_1q_1\le\varepsilon,0<k_2q_2\le\varepsilon}\sqrt{1-r(\mathbf{kq})}\exp\left(-\frac{(1-r(\mathbf{kq}))u_{\mathbf{T}}^2}{2(1+r(\mathbf{kq}))}\right)\\
&\le C(\log T_1T_2)^{-1/2}\sum_{0<k_1q_1\le\varepsilon,0<k_2q_2\le\varepsilon}[(kq_1)^{\alpha_1}+(kq_2)^{\alpha_2}]^{1/2}\exp\left(-\frac{1}{4}[(kq_1)^{\alpha_1}+(kq_2)^{\alpha_2}]\log(T_1T_2)\right)\\
&\le C(\log T_1T_2)^{-1/2}\sum_{0<k_1q_1\le\varepsilon,0<k_2q_2\le\varepsilon}\exp\left(-\frac{1}{4}[(kq_1)^{\alpha_1}+(kq_2)^{\alpha_2}]\log(T_1T_2)\right)\\
&\le C(\log T_1T_2)^{-1/2}\sum_{k_1=1}^{\infty}e^{-\frac{1}{4}(k_1\gamma_1)^{\alpha_1}}\sum_{k_2=1}^{\infty}e^{-\frac{1}{4}(k_2\gamma_2)^{\alpha_2}}\\
&\le C(\log T_1T_2)^{-1/2},
\end{aligned} \tag{28}$$

which shows $J_{\mathbf{T},1}\to 0$ as $\mathbf{T}\to\infty$.

Using the fact that $u_{\mathbf{T}}\sim(2\log T_1T_2)^{1/2}$, we obtain

$$\begin{aligned}
J_{\mathbf{T},2} &\le C\sum_{\substack{\mathbf{kq},\mathbf{lq}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lq},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1q_1-k_1q_1|,|l_2q_2-k_2q_2|\}>\varepsilon}}|r(\mathbf{kq},\mathbf{lq})-\varrho(\mathbf{kq},\mathbf{lq})|\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\varpi(\mathbf{kq},\mathbf{lq})}\right)\\
&\le C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{\substack{0\le k_1q_1\le T_1^a,0\le k_2q_2\le T_2^a,\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{k_1q_1,k_2q_2\}>\varepsilon}}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\\
&\le C\frac{T_1}{q_1}\frac{T_2}{q_2}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\sum_{0\le k_1q_1\le T_1^a,0\le k_2q_2\le T_2^a}1
\end{aligned}$$

$$\begin{aligned}
&\leq \quad C\frac{T_1}{q_1}\frac{T_2}{q_2}(T_1T_2)^{-\frac{2}{1+\vartheta(\varepsilon,\varepsilon)}} \sum_{0\leq k_1q_1\leq T_1^a, 0\leq k_2q_2\leq T_2^a} 1 \\
&\leq \quad C(T_1T_2)^{a-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{2/\alpha_1+2/\alpha_2}. \qquad (29)
\end{aligned}$$

Thus, $J_{\mathbf{T},2} \to 0$ as $\mathbf{T} \to \infty$ since $a < \frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$.

Second, we deal with the case that $\mathbf{kq} \in \mathbf{O_i}$ and $\mathbf{lq} \in \mathbf{O_j}$, $\mathbf{i} \neq \mathbf{j}$. Note that in this case, the distance between the points in any two rectangles $\mathbf{O_i}$ and $\mathbf{O_j}$ is large than T_1^b or T_2^b and $\varrho(\mathbf{kq},\mathbf{lq}) = \rho(\mathbf{T})$ for $\mathbf{kq} \in \mathbf{O_i}$ and $\mathbf{lq} \in \mathbf{O_j}$, $\mathbf{i} \neq \mathbf{j}$. Obviously, the sum in (24) is smaller than

$$C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j}\\ \mathbf{kq}\neq\mathbf{lq},1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}}} |r(\mathbf{kq},\mathbf{lq}) - \rho(\mathbf{T})| \exp\left(-\frac{u_{\mathbf{T}}^2}{1+\varpi(\mathbf{kq},\mathbf{lq})}\right). \qquad (30)$$

Split the sum of (30) into three parts, the first for $|k_1q_1 - l_1q_1| > 0$ and $|k_2q_2 - l_2q_2| > 0$, the second for $k_1q_1 - l_1q_1 = 0$ and $|k_2q_2 - l_2q_2| > 0$, the third for $k_2q_2 - l_2q_2 = 0$ and $|k_1q_1 - l_1q_1| > 0$ and denote them by $S_{\mathbf{T},i}$, $i = 1, 2, 3$, respectively. Let β be such that $0 < b < a < \beta < \frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$ for all sufficiently large $\mathbf{T}$.

We consider the term $S_{\mathbf{T},1}$ and split it into two parts as

$$S_{\mathbf{T},1} = C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lq},1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|\leq(T_1T_2)^\beta}} + C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lq},1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}} =: S_{\mathbf{T},11} + S_{\mathbf{T},12}.$$

For $S_{\mathbf{T},11}$, with the similar derivation as for (29), we have

$$\begin{aligned}
S_{\mathbf{T},11} \quad &\leq \quad C\frac{T_1}{q_1}\frac{T_2}{q_2} \sum_{\substack{0\leq k_1q_1\leq T_1, 0\leq k_2q_2\leq T_2,\\ k_1q_1k_2q_2\leq(T_1T_2)^\beta}} \exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right) \\
&\leq \quad C(T_1T_2)^{\beta-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{2/\alpha_1+2/\alpha_2}. \qquad (31)
\end{aligned}$$

Consequently, since $\beta < \frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$, we have $S_{\mathbf{T},11} \to 0$ as $\mathbf{T} \to \infty$.

For $S_{\mathbf{T},12}$, we need more precise estimation. Let's define

$$\omega_1(\mathbf{t}) = \max\{|r(\mathbf{t})|, |\rho(\mathbf{T})|\}$$

and

$$\theta_1(\mathbf{z}) = \sup_{\substack{\mathbf{0}\leq\mathbf{t}\leq\mathbf{T},\\ |t_1t_2|>z_1z_2}} \{\omega_1(\mathbf{t})\}.$$

By the Assumption **A3**, there exist constants $C > 0$ and $K > 0$ such that

$$\theta_1(\mathbf{t}) \log(t_1t_2) \leqslant K$$

for all $\mathbf{T}$ sufficiently large and $\mathbf{t}$ satisfying $t_1t_2 \geq C$. Thus for all $\mathbf{T}$ large enough and for (k_1q_1, k_2q_2) such that $k_1q_1k_2q_2 \geq (T_1T_2)^\beta$, $\theta_1(\mathbf{kq}) \leq K/\log(T_1T_2)^\beta$. Now making use of (27), we obtain

$$\begin{aligned}
&\frac{(T_1T_2)^2}{q_1^2q_2^2\log(T_1T_2)} \exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta, T_2^\beta)}\right) \\
&\leq \frac{(T_1T_2)^2}{q_1^2q_2^2\log(T_1T_2)} \exp\left(-\frac{u_{\mathbf{T}}^2}{1+K/\log(T_1T_2)^\beta}\right) \\
&\sim \frac{(T_1T_2)^2}{q_1^2q_2^2\log(T_1T_2)} \left((T_1T_2)^{-2}(\log T_1T_2)(\log T_1T_2)^{-(2/\alpha_1+2/\alpha_2)}\right)^{\frac{1}{1+K/\log(T_1T_2)^\beta}} \\
&\leq O(1)(T_1T_2)^{(2K/\log(T_1T_2)^\beta)/(1+K/\log(T_1T_2)^\beta)}(\log T_1T_2)^{((2/\alpha_1+2/\alpha_2-1)K/\log(T_1T_2)^\beta)/(1+K/\log(T_1T_2)^\beta)} \\
&= O(1). \qquad (32)
\end{aligned}$$

Therefore, by a similar argument as for the proof of Lemma 6.4.1 of Leadbetter et al. (1983) we obtain

$$
\begin{aligned}
S_{\mathbf{T},12} &\leq C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}|r(\mathbf{kq})-\rho(\mathbf{T})|\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta,T_2^\beta)}\right)\\
&\leq C\frac{T_1}{q_1}\frac{T_2}{q_2}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta,T_2^\beta)}\right)\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}|r(\mathbf{kq})-\rho(\mathbf{T})|\\
&= C\frac{(T_1T_2)^2}{q_1^2q_2^2\log(T_1T_2)}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta,T_2^\beta)}\right)\cdot\frac{q_1q_2\log(T_1T_2)}{T_1T_2}\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}|r(\mathbf{kq})-\rho(\mathbf{T})|\\
&\leq C\frac{q_1q_2\log(T_1T_2)}{T_1T_2}\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}|r(\mathbf{kq})-\rho(\mathbf{T})|\\
&\leq C\frac{q_1q_2}{\beta T_1T_2}\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}|r(\mathbf{kq})\log(k_1q_1k_2q_2)-r|\\
&\quad +Cr\frac{q_1q_2}{T_1T_2}\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}\left|1-\frac{\log(T_1T_2)}{\log(k_1q_1k_2q_2)}\right|. \qquad (33)
\end{aligned}
$$

By Assumption **A3**, the first term on the right-hand-side of (33) tends to 0 as $\mathbf{T}\to\infty$. Furthermore, the second term of the right-hand-side of (33) also tends to 0 by an integral estimate as follows (see also the proof of Lemma 6.4.1 of Leadbetter et al. (1983))

$$
\begin{aligned}
&\frac{q_1q_2}{T_1T_2}\sum_{\substack{\mathbf{0}\leq\mathbf{kq}\leq\mathbf{T},\mathbf{kq}\neq\mathbf{0}\\ k_1q_1k_2q_2>(T_1T_2)^\beta}}\left|1-\frac{\log(T_1T_2)}{\log(k_1q_1k_2q_2)}\right|\\
&\leq\frac{r}{\log(T_1T_2)^\beta}\frac{q_1q_2}{T_1T_2}\sum|\log(k_1q_1k_2q_2)-\log(T_1T_2)|\\
&=\frac{r}{\log(T_1T_2)^\beta}\frac{q_1q_2}{T_1T_2}\sum\left|\log\left(\frac{k_1q_1k_2q_2}{T_1T_2}\right)\right|\\
&=O\left(\frac{r}{\log(T_1T_2)^\beta}\int_0^1\int_0^1\log(xy)dxdy\right),
\end{aligned}
$$

which shows that $S_{\mathbf{T},12}\to 0$ as $\mathbf{T}\to\infty$. Thus, $S_{\mathbf{T},1}\to 0$ as $\mathbf{T}\to\infty$.
We consider the term $S_{\mathbf{T},2}$ and we will discuss it for two cases, the first for $(T_1T_2)^\beta > T_2$, and the second for $(T_1T_2)^\beta\leq T_2$.
For the case $(T_1T_2)^\beta > T_2$, by the same arguments as for (29), we have

$$
\begin{aligned}
S_{\mathbf{T},2} &= C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{0\leq k_2q_2\leq T_2,k_1q_1=0}|r(0,k_2q_2)-\rho(\mathbf{T})|\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(T_1^b,T_2^b)}\right)\\
&\leq C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{0\leq k_2q_2\leq T_2,k_1q_1=0}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\\
&\leq C(T_1T_2)^{\beta-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{1/\alpha_1+2/\alpha_2}.
\end{aligned}
$$

Therefore, $S_{\mathbf{T},2}\to 0$ as $\mathbf{T}\to\infty$ in view of $\beta<\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$.
For the second case $(T_1T_2)^\beta\leq T_2$, split $S_{\mathbf{T},2}$ into two parts as

$$
S_{\mathbf{T},2}=C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lq},\mathbf{1}\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ 0<|k_2q_2-l_2q_2|\leq(T_1T_2)^\beta,k_1q_1=l_1q_1}}+C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lq}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lq},\mathbf{1}\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ (T_1T_2)^\beta<|k_2q_2-l_2q_2|\leq T_2,k_1q_1=l_1q_1}}=:S_{\mathbf{T},21}+S_{\mathbf{T},22}.
$$

For $S_{\mathbf{T},21}$, similarly to the derivation of (29) again, we have

$$
S_{\mathbf{T},21}\leq C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{0\leq k_2q_2\leq(T_1T_2)^\beta,k_1q_1=0}|r(0,k_2q_2)-\rho(\mathbf{T})|\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(T_1^b,T_2^b)}\right)
$$

$$
\begin{aligned}
&\leq C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{0\leq k_2q_2\leq (T_1T_2)^\beta, k_1q_1=0}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\\
&\leq C(T_1T_2)^{\beta-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{1/\alpha_1+2/\alpha_2},
\end{aligned}
$$

which shows that $S_{\mathbf{T},21}\to 0$ as $\mathbf{T}\to\infty$.

For bound the term $S_{\mathbf{T},22}$, we need to define

$$
\omega_2(\mathbf{t})=\max\{|r(0,t_2)|,|\rho(\mathbf{T})|\}
$$

and

$$
\theta_2(\mathbf{z})=\sup_{\substack{\mathbf{0}\leq\mathbf{t}\leq\mathbf{T},\\ |t_2|>z_1z_2}}\{\omega(\mathbf{t})\}.
$$

By Assumption **A3** again, we have also $\theta_2(\mathbf{kq})\leq K/\log(T_1T_2)^\beta$ and $r(0,k_2q_2)\log(T_1T_2)\leq C$ for $k_1q_1=0$ and $k_2q_2>(T_1T_2)^\beta$. So by the same arguments as for (32), we have

$$
\frac{(T_1T_2)^2}{q_1^2q_2^2\log(T_1T_2)}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_2(T_1^\beta,T_2^\beta)}\right)=O(1)
$$

and we thus have

$$
\begin{aligned}
S_{\mathbf{T},22}&\leq C\frac{T_1}{q_1}\frac{T_2}{q_2}\sum_{(T_1T_2)^\beta<k_2q_2\leq T_2, k_1q_1=0}|r(0,k_2q_2)-\rho(\mathbf{T})|\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_2(T_1^\beta,T_2^\beta)}\right)\\
&= C\frac{(T_1T_2)^2}{q_1^2q_2^2\log(T_1T_2)}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\theta_2(T_1^\beta,T_2^\beta)}\right)\cdot\frac{q_1q_2\log(T_1T_2)}{T_1T_2}\sum_{(T_1T_2)^\beta<k_2q_2\leq T_2, k_1q_1=0}|r(0,k_2q_2)-\rho(\mathbf{T})|\\
&\leq C\frac{q_1q_2\log(T_1T_2)}{T_1T_2}\sum_{(T_1T_2)^\beta<k_2q_2\leq T_2, k_1q_1=0}|r(0,k_2q_2)-\rho(\mathbf{T})|\\
&\leq C\frac{q_1q_2\log(T_1T_2)}{T_1T_2}\sum_{(T_1T_2)^\beta<k_2q_2\leq T_2, k_1q_1=0}(|r(0,k_2q_2)|+\rho(\mathbf{T}))\\
&\leq C\frac{q_1q_2\log(T_1T_2)}{T_1T_2}\frac{T_2}{q_2}\frac{1}{\log(T_1T_2)}\\
&= C\frac{q_1}{T_1},
\end{aligned}
$$

which implies $S_{\mathbf{T},22}\to 0$ as $\mathbf{T}\to\infty$. Thus we have proved that $S_{\mathbf{T},2}\to 0$ as $\mathbf{T}\to\infty$. By the same arguments, we can show that $S_{\mathbf{T},3}\to 0$ as $\mathbf{T}\to\infty$. The proof of the lemma is complete. □

Lemma B2. *Under the conditions of Lemma 3.3, we have*

$$
\sum_{\substack{\mathbf{kp}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j}\\ \mathbf{kp}\neq\mathbf{lp},\mathbf{1\leq i,j\leq n}}}|r(\mathbf{kp},\mathbf{lp})-\varrho(\mathbf{kp},\mathbf{lp})|\int_0^1\frac{1}{\sqrt{1-r^{(h)}(\mathbf{kp},\mathbf{lp})}}\exp\left(-\frac{u_{\mathbf{T}}'^2}{1+r^{(h)}(\mathbf{kp},\mathbf{lp})}\right)dh\to 0 \tag{34}
$$

as $\mathbf{T}\to\infty$.

Proof: The proof is the same as that of Lemma B1, we omit the details.

Lemma B3. *Under the conditions of Lemma 3.3, we have*

$$
\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j}\\ \mathbf{kq}\neq\mathbf{lp},\mathbf{1\leq i,j\leq n}}}|r(\mathbf{kq},\mathbf{lp})-\varrho(\mathbf{kq},\mathbf{lp})|\int_0^1\frac{1}{\sqrt{1-r^{(h)}(\mathbf{kq},\mathbf{lp})}}\exp\left(-\frac{u_{\mathbf{T}}'^2+u_{\mathbf{T}}^2}{2(1+r^{(h)}(\mathbf{kq},\mathbf{lp}))}\right)dh\to 0 \tag{35}
$$

as $\mathbf{T}\to\infty$.

Proof: Recall that $\mathfrak{R}(p_i)$, $i=1,2$ can be sparse grids or Pickands grids. First, we consider the case that $\mathbf{kq},\mathbf{lp}$ in the same interval $\mathbf{O_i}$. Split the sum in (35) into two parts as

$$\sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon}} + \sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}>\varepsilon}} =: W_{\mathbf{T},1}+W_{\mathbf{T},2}. \tag{36}$$

We deal with $W_{\mathbf{T},1}$. For $\mathbf{kq},\mathbf{lp}$ in the same interval $\mathbf{O_i}$, we have $\varrho(\mathbf{kq},\mathbf{lp})-r(\mathbf{kq},\mathbf{lq})=\rho(\mathbf{T})(1-r(\mathbf{kq},\mathbf{lq}))$. By Assumption **A1** we can also choose small enough $\varepsilon>0$ such that $\varrho(\mathbf{kq},\mathbf{lp})=r(\mathbf{kq},\mathbf{lp})+(1-r(\mathbf{kq},\mathbf{lp}))\rho(\mathbf{T})\sim r(\mathbf{kq},\mathbf{lp})$ for sufficiently large $\mathbf{T}$ and $\max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon$. By the definitions of $u_{\mathbf{T}}$ and $u'_{\mathbf{T}}$, we have

$$v_{\mathbf{T}}^2 := \frac{1}{2}(u_{\mathbf{T}}^2+(u'_{\mathbf{T}})^2) = 2\log(T_1T_2)-\log\log(T_1T_2)+\log(p_1^{-1}p_2^{-1})+(1/\alpha_1+1/\alpha_2)\log\log(T_1T_2)+O(1). \tag{37}$$

Consequently, in view of (26), we obtain

$$\begin{aligned}
W_{\mathbf{T},1} &\le C \sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon}} |r(\mathbf{kq},\mathbf{lp})-\varrho(\mathbf{kq},\mathbf{lp})|\frac{1}{\sqrt{1-r(\mathbf{kq},\mathbf{lp})}}\exp\left(-\frac{v_{\mathbf{T}}^2}{1+r(\mathbf{kq},\mathbf{lq})}\right)\\
&\le C\rho(\mathbf{T}) \sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon}} \sqrt{1-r(\mathbf{kq},\mathbf{lp})}\exp\left(-\frac{v_{\mathbf{T}}^2}{2}\right)\exp\left(-\frac{(1-r(\mathbf{kq},\mathbf{lp}))v_{\mathbf{T}}^2}{(1+r(\mathbf{kq},\mathbf{lp}))}\right)\\
&\le C\rho(\mathbf{T})(T_1T_2)^{-1}(p_1p_2)^{1/2}(\log T_1T_2)^{1/2-1/2\alpha_1-1/2\alpha_2}\times\\
&\qquad \sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon}} \sqrt{1-r(\mathbf{kq}-\mathbf{lp})}\exp\left(-\frac{(1-r(\mathbf{kq}-\mathbf{lp}))u_{\mathbf{T}}^2}{2(1+r(\mathbf{kq}-\mathbf{lp}))}\right)\\
&\le C(T_1T_2)^{-1}(p_1p_2)^{1/2}(\log T_1T_2)^{-1/2-1/2\alpha_1-1/2\alpha_2}\times\\
&\qquad \sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon}} \sqrt{|l_1p_1-k_1q_1|^{\alpha_1}+|l_2p_2-k_2q_2|^{\alpha_2}}\times\\
&\qquad \exp\left(-\frac{(|l_1p_1-k_1q_1|^{\alpha_1}+|l_2p_2-k_2q_2|^{\alpha_2})v_{\mathbf{T}}^2}{8}\right).
\end{aligned} \tag{38}$$

Noting that $q_i=\gamma_i\log(T_1T_2)^{1/\alpha_i}$ and $\mathfrak{R}(p_i)$, $i=1,2$ are sparse grids or Pickands grids, a direct calculation shows that

$$\begin{aligned}
&\sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}\le\varepsilon}} \sqrt{|l_1p_1-k_1q_1|^{\alpha_1}+|l_2p_2-k_2q_2|^{\alpha_2}}\times\\
&\qquad \exp\left(-\frac{(|l_1p_1-k_1q_1|^{\alpha_1}+|l_2p_2-k_2q_2|^{\alpha_2})v_{\mathbf{T}}^2}{8}\right)\\
&\le CT_1T_2p_1^{-1}p_2^{-1}\sum_{0<k_1q_1<\varepsilon,0<k_2q_2<\varepsilon}\exp\left(-\frac{1}{4}[(k_1q_1)^{\alpha_1}+(k_2q_2)^{\alpha_2}]\log(T_1T_2)\right)\\
&\le CT_1T_2p_1^{-1}p_2^{-1}\sum_{k_1=1}^{\infty}e^{-\frac{1}{4}(k_1\gamma_1)^{\alpha_1}}\sum_{k_2=1}^{\infty}e^{-\frac{1}{4}(k_2\gamma_2)^{\alpha_2}}\\
&\le CT_1T_2p_1^{-1}p_2^{-1},
\end{aligned}$$

which combine with (38) shows that $W_{\mathbf{T},1}\to 0$ as $\mathbf{T}\to\infty$.
Using the fact that $v_{\mathbf{T}}\sim(2\log T_1T_2)^{1/2}$, we obtain

$$\begin{aligned}
W_{\mathbf{T},2} &\le C\sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}>\varepsilon}} |r(\mathbf{kq},\mathbf{lp})-\varrho(\mathbf{kq},\mathbf{lp})|\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\varpi(\mathbf{kq},\mathbf{lp})}\right)\\
&\le C\sum_{\substack{\mathbf{kq},\mathbf{lp}\in\mathbf{O_i},\mathbf{kq}\neq\mathbf{lp},\mathbf{i}=\mathbf{1},\cdots,\mathbf{n},\\ \max\{|l_1p_1-k_1q_1|,|l_2p_2-k_2q_2|\}>\varepsilon}} \exp\left(-\frac{v_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\\
&\le C\frac{T_1}{p_1}\frac{T_2}{p_2}\exp\left(-\frac{u_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\sum_{0\le k_1q_1\le T_1^a,0\le k_2q_2\le T_2^a}1
\end{aligned}$$

$$\leq \quad C\frac{T_1}{p_1}\frac{T_2}{p_2}(T_1T_2)^{-\frac{2}{1+\vartheta(\varepsilon,\varepsilon)}} \sum_{0\leq k_1q_1\leq T_1^a, 0\leq k_2q_2\leq T_2^a} 1$$

$$\leq \quad C(T_1T_2)^{a-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{1/\alpha_1+1/\alpha_2}(p_1p_2)^{-1}. \tag{39}$$

Thus, $W_{\mathbf{T},2} \to 0$ as $\mathbf{T} \to \infty$ by virtue of $a < \frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$ again.
Second, we deal with the case that $\mathbf{kq} \in \mathbf{O_i}$ and $\mathbf{lp} \in \mathbf{O_j}$, $\mathbf{i} \neq \mathbf{j}$. Note that in this case, the distance between the points in any two rectangles $\mathbf{O_i}$ and $\mathbf{O_j}$ is large than T_1^b or T_2^b and $\varrho(\mathbf{kq}, \mathbf{lp}) = \rho(\mathbf{T})$ for $\mathbf{kq} \in \mathbf{O_i}$ and $\mathbf{lp} \in \mathbf{O_j}$, $\mathbf{i} \neq \mathbf{j}$. Obviously, the sum in (35) is at most

$$C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i}, \mathbf{lp}\in\mathbf{O_j} \\ \mathbf{kq}\neq\mathbf{lp}, 1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}}} |r(\mathbf{kq}, \mathbf{lp}) - \rho(\mathbf{T})| \exp\left(-\frac{v_{\mathbf{T}}^2}{1+\varpi(\mathbf{kq}, \mathbf{lp})}\right). \tag{40}$$

Split the sum of (40) into three parts, the first for $|k_1q_1 - l_1p_1| > 0$ and $|k_2q_2 - l_2p_2| > 0$, the second for $k_1q_1 - l_1p_1 = 0$ and $|k_2q_2 - l_2p_2| > 0$, the third for $k_2q_2 - l_2p_2 = 0$ and $|k_1q_1 - l_1p_1| > 0$ and denote them by $M_{\mathbf{T},i}$, $i = 1, 2, 3$, respectively. Let β be chosen as before, ie, $0 < b < a < \beta < \frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$ for all sufficiently large $\mathbf{T}$.
We consider the term $M_{\mathbf{T},1}$ and split it into two parts as

$$M_{\mathbf{T},1} = C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i}, \mathbf{lp}\in\mathbf{O_j}, \mathbf{kq}\neq\mathbf{lp}, 1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n} \\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|\leq(T_1T_2)^\beta}} + C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i}, \mathbf{lp}\in\mathbf{O_j}, \mathbf{kq}\neq\mathbf{lp}, 1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n} \\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}} =: M_{\mathbf{T},11} + M_{\mathbf{T},12}.$$

For $M_{\mathbf{T},11}$, with the similar derivation as for (39), we have

$$M_{\mathbf{T},11} \quad \leq \quad C\frac{T_1}{p_1}\frac{T_2}{p_2} \sum_{\substack{0\leq k_1q_1\leq T_1, 0\leq k_2q_2\leq T_2, \\ k_1q_1k_2q_2\leq(T_1T_2)^\beta}} \exp\left(-\frac{v_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)$$

$$\leq \quad C(T_1T_2)^{\beta-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{1/\alpha_1+1/\alpha_2}(p_1p_2)^{-1}. \tag{41}$$

Thus, $M_{\mathbf{T},11} \to 0$ as $\mathbf{T} \to \infty$, since $\beta < \frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}$.
For $M_{\mathbf{T},12}$, we need also more precise estimation. Recall that

$$\omega_1(\mathbf{t}) = \max\{|r(\mathbf{t})|, |\rho(\mathbf{T})|\} \quad \text{and} \quad \theta_1(\mathbf{z}) = \sup_{\substack{\mathbf{0}\leq\mathbf{t}\leq\mathbf{T}, \\ |t_1t_2|>z_1z_2}} \{\omega_1(\mathbf{t})\}.$$

Now using (37) again, by the same arguments as for (32), we obtain

$$\begin{aligned}
&\frac{(T_1T_2)^2}{q_1q_2p_1p_2\log(T_1T_2)} \exp\left(-\frac{v_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta, T_2^\beta)}\right) \\
&\leq \frac{(T_1T_2)^2}{q_1q_2p_1p_2\log(T_1T_2)} \exp\left(-\frac{v_{\mathbf{T}}^2}{1+K/\log(T_1T_2)^\beta}\right) \\
&\sim \frac{(T_1T_2)^2}{q_1q_2p_1p_2\log(T_1T_2)} \left((T_1T_2)^{-2}(\log T_1T_2)(\log T_1T_2)^{-(1/\alpha_1+1/\alpha_2)}(p_1p_2)^{-1}\right)^{\frac{1}{1+K/\log(T_1T_2)^\beta}} \\
&= O(1)
\end{aligned} \tag{42}$$

and then we thus have

$$\begin{aligned}
M_{\mathbf{T},12} \quad &\leq \quad C \sum_{\substack{\mathbf{kq}\in\mathbf{O_i}, \mathbf{lp}\in\mathbf{O_j}, \mathbf{kq}\neq\mathbf{lp}, 1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n} \\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}} |r(\mathbf{kq}-\mathbf{lp}) - \rho(\mathbf{T})| \exp\left(-\frac{v_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta, T_2^\beta)}\right) \\
&= \quad C\frac{(T_1T_2)^2}{q_1q_2p_1p_2\log(T_1T_2)} \exp\left(-\frac{v_{\mathbf{T}}^2}{1+\theta_1(T_1^\beta, T_2^\beta)}\right) \times \\
&\qquad \frac{q_1q_2p_1p_2\log(T_1T_2)}{(T_1T_2)^2} \sum_{\substack{\mathbf{kq}\in\mathbf{O_i}, \mathbf{lp}\in\mathbf{O_j}, \mathbf{kq}\neq\mathbf{lp}, 1\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n} \\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}} |r(\mathbf{kq}-\mathbf{lp}) - \rho(\mathbf{T})|
\end{aligned}$$

$$
\begin{aligned}
&\le C\frac{q_1q_2p_1p_2\log(T_1T_2)}{(T_1T_2)^2}\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}}|r(\mathbf{kq}-\mathbf{lp})-\rho(\mathbf{T})|\\
&\le C\frac{q_1q_2p_1p_2}{\beta(T_1T_2)^2}\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}}|r(\mathbf{kq}-\mathbf{lp})\log((k_1q_1-l_1p_1)(k_2q_2-l_2p_2))-r|\\
&\quad+Cr\frac{q_1q_2p_1p_2}{(T_1T_2)^2}\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ |k_1q_1-l_1q_1||k_2q_2-l_2q_2|>(T_1T_2)^\beta}}\left|1-\frac{\log(T_1T_2)}{\log((k_1q_1-l_1p_1)(k_2q_2-l_2p_2))}\right|. \qquad (43)
\end{aligned}
$$

By Assumption **A3**, the first term on the right-hand-side of (43) tends to 0 as $\mathbf{T}\to\infty$. Furthermore, the second term of the right-hand-side of (43) also tends to 0 by an integral estimate as for Lemma B1. Thus $M_{\mathbf{T},12}\to 0$ as $\mathbf{T}\to\infty$ and then $M_{\mathbf{T},1}\to 0$ as $\mathbf{T}\to\infty$.

We consider the term $M_{\mathbf{T},2}$ now. As the the proof of the previous lemma, we also discuss it for two cases, the first for $(T_1T_2)^\beta > T_2$, and the second for $(T_1T_2)^\beta\le T_2$.

For the case $(T_1T_2)^\beta > T_2$, by the same arguments as for (39), we have

$$
\begin{aligned}
M_{\mathbf{T},2} &= C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})|\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\vartheta(T_1^b,T_2^b)}\right)\\
&\le C\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})|\\
&\le C(T_1T_2)^{\beta-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{1/\alpha_2}(p_1p_2)^{-1},
\end{aligned}
$$

which shows that $M_{\mathbf{T},2}\to 0$ as $\mathbf{T}\to\infty$, where in the last step, we use the fact that the number of (k_1,l_1) such that $k_1q_1-l_1p_1=0$ does not exceed T_1/p_1 and $(T_1T_2)^\beta > T_2$.

For the second case $(T_1T_2)^\beta\le T_2$, split $M_{\mathbf{T},2}$ into two parts as

$$
M_{\mathbf{T},2}=C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ 0<|k_2q_2-l_2p_2|\le(T_1T_2)^\beta,k_1q_1=l_1p_1}} +C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ (T_1T_2)^\beta<|k_2q_2-l_2p_2|\le T_2,k_1q_1=l_1p_1}} =: M_{\mathbf{T},21}+M_{\mathbf{T},22}.
$$

For $M_{\mathbf{T},21}$, similarly to the derivation of (39) again, we have

$$
\begin{aligned}
M_{\mathbf{T},21} &\le C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ 0<|k_2q_2-l_2p_2|\le(T_1T_2)^\beta,k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})|\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\vartheta(T_1^b,T_2^b)}\right)\\
&\le C\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\vartheta(\varepsilon,\varepsilon)}\right)\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ 0<|k_2q_2-l_2p_2|\le(T_1T_2)^\beta,k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})|\\
&\le C(T_1T_2)^{\beta-\frac{1-\vartheta(\varepsilon,\varepsilon)}{1+\vartheta(\varepsilon,\varepsilon)}}(\log T_1T_2)^{1/\alpha_2}(p_1p_2)^{-1},
\end{aligned}
$$

which shows that $M_{\mathbf{T},21}\to 0$ as $\mathbf{T}\to\infty$.

For $M_{\mathbf{T},22}$, we recall that

$$
\omega_2(\mathbf{t})=\max\{|r(0,t_2)|,|\rho(\mathbf{T})|\}\ \text{ and }\ \theta_2(\mathbf{z})=\sup_{\substack{\mathbf{0}\le\mathbf{t}\le\mathbf{T},\\ |t_2|>z_1z_2}}\{\omega(\mathbf{t})\}.
$$

So by the same arguments as for (32), we have

$$
\frac{(T_1T_2)^2}{q_1q_2p_1p_2\log(T_1T_2)}\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\theta_2(T_1^\beta,T_2^\beta)}\right)=O(1)
$$

and we thus have

$$
M_{\mathbf{T},22}\le C\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},1\le\mathbf{i}\neq\mathbf{j}\le\mathbf{n}\\ (T_1T_2)^\beta<|k_2q_2-l_2p_2|,k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})|\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\theta_2(T_1^\beta,T_2^\beta)}\right)
$$

$$
\begin{aligned}
&= C\frac{(T_1T_2)^2}{q_1q_2p_1p_2\log(T_1T_2)}\exp\left(-\frac{v_{\mathbf{T}}^2}{1+\theta_2(T_1^\beta,T_2^\beta)}\right)\times \\
&\quad \frac{q_1q_2p_1p_2\log(T_1T_2)}{(T_1T_2)^2}\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},\mathbf{1}\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ (T_1T_2)^\beta<|k_2q_2-l_2p_2|,k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})| \\
&\leq C\frac{q_1q_2p_1p_2\log(T_1T_2)}{(T_1T_2)^2}\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},\mathbf{1}\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ (T_1T_2)^\beta<|k_2q_2-l_2p_2|,k_1q_1-l_1p_1=0}}|r(0,k_2q_2-l_2p_2)-\rho(\mathbf{T})| \\
&\leq C\frac{q_1q_2p_1p_2\log(T_1T_2)}{(T_1T_2)^2}\sum_{\substack{\mathbf{kq}\in\mathbf{O_i},\mathbf{lp}\in\mathbf{O_j},\mathbf{kq}\neq\mathbf{lp},\mathbf{1}\leq\mathbf{i}\neq\mathbf{j}\leq\mathbf{n}\\ (T_1T_2)^\beta<|k_2q_2-l_2p_2|,k_1q_1-l_1p_1=0}}(|r(0,k_2q_2-l_2p_2)|+\rho(\mathbf{T})) \\
&\leq C\frac{q_1q_2p_1p_2\log(T_1T_2)}{(T_1T_2)^2}\frac{T_2^2T_1}{p_1q_2p_2}\frac{1}{\log(T_1T_2)} \\
&= C\frac{q_1}{T_1},
\end{aligned}
$$

which implies $M_{\mathbf{T},22}\to 0$ as $\mathbf{T}\to\infty$. Now we have showed that $M_{\mathbf{T},2}\to 0$ as $\mathbf{T}\to\infty$. By the same arguments, we can show that $M_{\mathbf{T},3}\to 0$ as $\mathbf{T}\to\infty$. The proof of the lemma is complete. □

Acknowledgement: We would like to thank the referee for several important suggestions and corrections.

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